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ON THE RIEMANN ZETA FUNCTION*

BY C. F. CRAIG

In this note a functional relation for the Riemann zeta function is established, and its relation to the well known formula given by Riemann is pointed out. It is believed that the relation is new.

Recall that the even elliptic theta constants are

$$(1)\qquad \begin{cases} \vartheta_{00}(x) = 1 + 2\sum_{n=1}^{\infty} q^{4n^2} = 1 + 2\phi_{00}(x), \\ \vartheta_{01}(x) = 2\sum_{n=1}^{\infty} q^{(2n-1)^2} = 2\phi_{01}(x), \\ \vartheta_{10}(x) = 1 - 2\sum_{n=1}^{\infty} (-1)^{n-1} q^{4n^2} = 1 - 2\phi_{10}(x), \end{cases}$$

with $q = e^{-(\pi/4)x^2}$. These series converge uniformly and absolutely for $0 < \delta \leqq x$. Furthermore

$$\vartheta_{00}\left(\frac{1}{x}\right) = x\vartheta_{00}(x), \qquad \vartheta_{01}\left(\frac{1}{x}\right) = x\vartheta_{10}(x), \qquad \vartheta_{10}\left(\frac{1}{x}\right) = x\vartheta_{01}(x).$$

Let

$$(2)\qquad \vartheta(x) = x^{1/2}[\vartheta_{01}(x) + \vartheta_{10}(x) - \vartheta_{00}(x)];$$

then

$$(3)\qquad \vartheta\left(\frac{1}{x}\right) = \vartheta(x).$$

Also

$$1 + 2\phi_{00}\left(\frac{1}{x}\right) = x[1 + 2\phi_{00}(x)]$$

with two similar relations for ϕ_{01} and ϕ_{10}. From these follow

$$(4)\qquad \begin{cases} \lim_{x\to 0} x\phi_{00}(x) = \frac{1}{2}, & \lim_{x\to 0} x\phi_{10}(x) = 0, \\ \lim_{x\to 0} x\phi_{01}(x) = \frac{1}{2}, & \lim_{x\to 0} x^h\,\vartheta(x) = 0, \end{cases}$$

where h is any constant. To prove the last of the expressions

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(4) observe that

$$\vartheta(y) = 2y^{1/2}\sum_1^\infty Q_n(y),$$

where $Q_n(y) = q^{(2n-1)^2} - 2q^{4(2n-1)^2}$. For $y \geqq 1$, $Q_n(y) > 0$, and hence by (3) $\vartheta(y)$ is never negative. Also

$$Q_n(y) < e^{-(\pi/4)y^2}e^{-n\pi}, \qquad y \geqq 1,$$

and hence

$$\vartheta(y) < 2y^{1/2}e^{-(\pi/4)y^2}(1 - e^{-\pi/4})^{-1}, \qquad y \geqq 1.$$

In $x^h\vartheta(x)$ placing $x = 1/y \leqq 1$, and using (3) together with the last inequality, we find

$$|x^h\vartheta(x)| < 2(1 - e^{-\pi/4})^{-1}|y^{1-h}|e^{-(\pi/4)y^2}.$$

The last of the expressions (4) follows for $y \to \infty$.

Recall further that

$$(5)\qquad \begin{cases} \zeta(s) = 1 + \dfrac{1}{2^s} + \dfrac{1}{3^s} + \dfrac{1}{4^s} + \cdots, \\ (1 - 2^{1-s})\zeta(s) = 1 - \dfrac{1}{2^s} + \dfrac{1}{3^s} - \dfrac{1}{4^s} + \cdots, \\ (1 - 2^{-s})\zeta(s) = 1 + \dfrac{1}{3^s} + \cdots, \end{cases}$$

and that each of these series converges absolutely and uniformly for $R(s) \geqq 1 + \delta$, $R(s)$ being the real part of s. In

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-z}z^{(s/2)-1}dz, \qquad R(s) > 0,$$

replace z by $k^2\pi x^2$ and there results

$$(6)\qquad \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)k^{-s} = 2\int_0^\infty e^{-k^2\pi x^2}x^{s-1}dx, \qquad R(s) > 0.$$

In (6) put $k = n$, multiply each side of the equality by $(-1)^{n-1}$, and sum with respect to n. This gives, on account of (5),

$$(1 - 2^{1-s})\chi(s) = \sum_{n=1}^\infty 2\int_0^\infty (-1)^{n-1}e^{-n^2\pi x^2}x^{s-1}dx, \qquad R(s) > 1,$$

where

$$\chi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

An easy investigation * shows that the operations of integration and summation are reversible, giving

$$(1 - 2^{1-s})\chi(s) = 2\int_0^\infty \phi_{10}(x)x^{s-1}dx, \qquad R(s) \geqq \delta + 2.$$

The use of (6) with $k = n$ and $2k = 2n - 1$ gives, on summing with respect to n,

$$\chi(s) = 2\int_0^\infty \phi_{00}(x)x^{s-1}dx, \qquad R(s) \geqq \delta + 2,$$

$$2^s(1 - 2^{-s})\chi(s) = 2\int_0^\infty \phi_{01}(x)x^{s-1}dx, \qquad R(s) \geqq \delta + 2.$$

The first of this pair of relations is Riemann's well known formula. These three equalities combine to yield, by using the relations (1) and (2),

$$(7) \qquad -(2^s - 1)(2^{1-s} - 1)\chi(s) = \int_0^\infty \vartheta(x)x^{s-\frac{1}{2}}x^{-1}dx.$$

The derivation of formula (7) is the principal object of this note. Thanks to the last of the relations (4), the integral in (7) converges for every value of s and hence the representation is valid everywhere. In (7) replace x with $1/x$, use (3) and obtain

$$-(2^s - 1)(2^{1-s} - 1)\chi(s) = \int_0^\infty \vartheta(x)x^{-(s-\frac{1}{2})}x^{-1}dx.$$

The addition of this equation to (7) gives

$$(8) \quad -2(2^s-1)(2^{1-s}-1)\chi(s) = \int_0^\infty \vartheta(x)[x^{s-\frac{1}{2}}+x^{-(s-\frac{1}{2})}]x^{-1}dx.$$

In this replace s by $1 - s$ and we deduce the well known property of $\chi(s)$, viz., $\chi(s) = \chi(1 - s)$. Since $\vartheta(x)$ is never

* That given by Whittaker and Watson, *Modern Analysis*, 3d ed., p. 273, is readily adapted.

negative, the right hand member of (7) does not vanish for s real. Thus $\chi(s)$ has a simple pole at $s = 0$ and at $s = 1$ and has no real zeros. Recalling the location of the poles, all simple, of $\Gamma(s/2)$, we observe that $\zeta(s)$ has $-2, -4, -6, \cdots$ as its only real zeros and that these are simple.

In (8), let $s = \frac{1}{2} + it$, and we have

$$(9) \qquad X(t) = \int_0^\infty \vartheta(x) \cos (t \log x) x^{-1} dx,$$

with

$$X(t) = (2^{\frac{1}{2}+it} - 1)(1 - 2^{\frac{1}{2}-it})\chi(\tfrac{1}{2} + it).$$

In (9), separate the integral into the two parts from 0 to 1 and from 1 to ∞, in the first replace x with $1/x$, employ (3), and we have

$$X(t) = 2 \int_1^\infty \vartheta(x) \cos (t \log x) x^{-1} dx.$$

Now replace x by e^x, and we obtain

$$X(t) = 2 \int_0^\infty \vartheta(e^x) \cos (tx) dx.$$

Observing that $X(t)$ is an entire function, we obtain the series representation

$$(10) \qquad X(t) = \sum_{n=0}^{\infty} b_n t^{2n}.$$

This series has the known zeros $t = (2k\pi/\log 2) \pm i/2$, where $k = 0, \pm 1, \pm 2, \cdots$, together with an unlimited number of real zeros. So far neither the series (10) nor any of the chain of integrals leading up to it has yielded any new information relative to the verification of the hypothesis of Riemann about the nature of the zeros of $\zeta(s)$. If we get rid of the zeros of $X(t)$ corresponding to those of $(1 - 2^s)(1 - 2^{1-s})$ by employing theta relations such as $2\vartheta_{01}(2x) = \vartheta_{00}(x) - \vartheta_{10}(x)$, then (7), or any of its consequences, reduces immediately to the corresponding result in the Riemann development.

CORNELL UNIVERSITY

NOTE ON A CERTAIN TYPE OF RULED SURFACE *

BY W. C. GRAUSTEIN

In the January, 1923, number of this BULLETIN, J. K. Whittemore discusses ruled surfaces having the property that any two secondary asymptotic lines cut equal segments from the rulings. It so happens that, in investigating the determination of a surface by the linear element of its spherical representation and its total and mean curvatures, the present writer was led to consider the same class of ruled surfaces, with the results which are set forth in this note. The method of attack differs from that of Whittemore and the facts obtained overlap only in the case of the characteristic property, namely, that the rulings are parallel to a plane and the parameter of distribution is constant.

Any two secondary asymptotic lines of a ruled surface cut equal segments from the rulings if and only if the surface, when referred to its asymptotic lines as the parametric curves, admits a representation of the usual form,

$$x_i = \xi_i(v) + u\eta_i(v), \qquad (i = 1, 2, 3), \tag{1}$$

where η_1, η_2, η_3 are the direction cosines of the ruling, η, and u is the algebraic distance along the ruling from the directrix, $\xi = \xi(v)$. Analytically, this condition amounts to demanding that $D'' = 0$. But HD'' is a quadratic polynomial in u, whose coefficients are functions of v alone. Thus the condition $D'' = 0$ gives rise to three equations, namely:

$$(\eta\xi'\xi'') = 0, \qquad (\eta\eta'\xi'') + (\eta\xi'\eta'') = 0, \qquad (\eta\eta'\eta'') = 0. \tag{2}$$

The vanishing of the last determinant is the condition that the director cone degenerate into a plane. The spherical indicatrix, $\eta_i = \eta_i(v)$, $(i = 1, 2, 3)$, is then a circle of unit radius, in particular, the circle in which the director plane cuts the unit sphere. If we choose as the parameter v the arc of this circle, it follows that $\eta_i'' = -\eta_i$, $(i = 1, 2, 3)$, and conditions (2) become

$$(\eta\xi'\xi'') = 0, \qquad (\eta\eta'\xi'') = 0. \tag{3}$$

* Presented to the Society, April 28, 1923.

The first of these conditions requires merely that the directrix be a secondary asymptotic line and can always be assumed fulfilled. The second is the condition that the parameter of distribution be constant, for the value of this parameter is readily shown to be $\pm\,(\eta\eta'\xi')$. The characteristic property is thus established.

If we exclude the trivial case of a developable, when the surface is necessarily a plane, the directions η, ξ', η' are distinct and not parallel to a plane. Consequently, equations (3) are satisfied only if η and ξ'' are linearly dependent, that is, only if *

$$\eta_i = \frac{\xi_i''}{\sqrt{\xi''|\xi''}}, \qquad (i = 1, 2, 3), \tag{4}$$

where there is no loss of generality in admitting merely the positive square root. Moreover, the assumption $(\xi''|\xi'') \neq 0$ is readily justified, since not all the secondary asymptotic lines can be straight and one which is not can be chosen as the directrix. If we introduce the curve y defined by the equations $y_i = \xi_i'$, $(i = 1, 2, 3)$, the representation (1) of the surface becomes

$$x_i = \int y_i dv + u\eta_i, \qquad (i = 1, 2, 3). \tag{5}$$

Furthermore, we have, in place of (4),

$$\eta_i = \frac{y_i'}{\sqrt{y'|y'}}, \qquad (i = 1, 2, 3).$$

In other words, the circle $\eta = \eta(v)$, which is the spherical indicatrix of the surface, is also the tangent indicatrix of the curve y. This curve, then, is a plane curve. Its plane is parallel to the director plane, but not coincident with it, since otherwise (5) would represent a developable (a plane).

If $y = y(v)$ is an arbitrary plane curve not in a plane through the origin and v is the arc of its tangent indicatrix, $\eta = \eta(v)$, the general ruled surface having the property that the secondary asymptotic lines cut equal segments from the rulings is represented by equations (5).

* If a denotes the number triple a_1, a_2, a_3, then $(a|a) = a_1^2 + a_2^2 + a_3^2$.

We next investigate under what conditions the surface is a conoid. The asymptotic line $u = u_0$ is straight only when the vector product of x_v and x_{vv} vanishes for $u = u_0$. But this vector product is equal to the scalar, $R - u$, multiplied by a non-vanishing triple, where R is the radius of curvature of the curve y. Hence it is zero for some constant value of u only if R is constant.

The surface is a conoid if and only if the curve y is a circle.

For $u = R =$ const., $x_{vv} = 0$. Hence the arc of the directrix line of the conoid can be taken proportional to v, or, since the arc v of the unit circle, $\eta = \eta(v)$, is also the corresponding angle at the center of the circle, the distance between two arbitrary points of the directrix line is proportional to the angle between the rulings through these points. In fact, if we introduce $u - R = \bar{u}$ as a new parameter, equations (5) become

$$x_i = a_i v + \bar{u}\eta_i(v), \qquad (i = 1, 2, 3), \tag{6}$$

where a_1, a_2, a_3 are constants defining a direction not parallel to the director plane.

The secondary asymptotic lines of a conoid cut equal segments from the rulings if and only if the angle through which a variable ruling turns is proportional to the distance along the directrix line through which the ruling slides.

A right conoid with this property is necessarily a right helicoid, the only minimal ruled surface. It is in this case alone that the curve y of (5) is a circle subtending at the origin a cone of revolution.

If the (x_1, x_2)-plane be taken as the director plane, η_1, η_2, η_3 can be chosen respectively as $\cos v$, $\sin v$, 0, and equations (5) become

$$x_1 = \int dv \int R \cos v \, dv,$$

$$x_2 = \int dv \int R \sin v \, dv,$$

$$x_3 = cv,$$

where R is an arbitrary function of v and c an arbitrary constant, not zero. Whittemore's equations (1″), taken in conjunction with his relation (6), are reducible to this form.

HARVARD UNIVERSITY

THE DIFFERENTIATION OF A FUNCTION OF A FUNCTION

BY H. S. CARSLAW

The reviewer of Rothe's interesting work *Vorlesungen über Höhere Mathematik* (1921) in this BULLETIN (vol. 28, p. 468) calls attention to the author's tentative claim that the first valid proof of the formula for the derivative of a function of a function is to be found therein, and he mentions the careful treatment of the question in Pierpont's *Theory of Functions of a Real Variable* (1905).

It is perhaps worth while to notice that Dini in his *Lezioni di Analisi Infinitesimale* (autographed edition, 1877) and Genocchi-Peano in their *Calcolo Differenziale* (1884) both gave satisfactory proofs. The treatment of Genocchi-Peano is cited and reproduced by Stolz in *Grundzüge der Differential- und Integralrechnung,* Bd. I (1893).

A proof on the same lines as that of Pierpont was given by Tannery in his *Introduction à la Théorie des Fonctions d'une Variable* (1886). See also Cesàro's *Lehrbuch der Algebraischen Analysis und der Infinitesimalrechnung,* Deutsch von Kowalewski (1904), and Kowalewski's *Grundzüge der Differential- und Integralrechnung* (1909).

It is remarkable that even the most careful English writers on the calculus have missed the defect in the proof to be found in our standard works on that subject.

THE UNIVERSITY OF SYDNEY, AUSTRALIA

ON THE RELATIVE CURVATURE OF TWO CURVES IN V_n*

BY JOSEPH LIPKA

1. *Definitions of Geodesic Curvature.* In any space of n dimensions V_n whose first fundamental form is given by †

$$ds^2 = \sum_{ik} a_{ik} dx_i dx_k, \tag{1}$$

the geodesic curvature κ of a curve c at a point P may be defined in one of the two following ways:

(i) Draw the geodesic g tangent to c at P and on g and c lay off from P equal infinitesimal arc lengths δs; let Q and $\overline{Q}$ be the extremities of these arcs on c and g respectively; then ‡

$$\kappa = \lim_{Q \to P} \frac{2Q\overline{Q}}{(\delta s)^2}\cdot \tag{2}$$

(ii) Consider an infinitesimal element $PQ = \delta s$ of c and the geodesic g having this element in common with c, i.e., as Q approaches P as a limit, c and g will approach tangency at P; the immediately following elements of c and g will not in general coincide but will form at Q an infinitesimal angle $\delta\omega$; then §

$$\kappa = \lim_{Q \to P} \frac{\delta\omega}{\delta s}\cdot \tag{3}$$

Both these definitions lead to the same analytical expression

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* Presented to the Society, February 24, 1923.

† Throughout this paper all summations extend from 1 to n for the indicated subscripts.

‡ This definition is that given by Voss, MATHEMATISCHE ANNALEN, vol. 16. Cf. Bianchi, *Geometria differenziale*, 2d ed., vol. 1, p. 363.

§ This definition, or an equivalent one in terms of the parallelism of Levi-Civita, was given by the author in a paper, *Sulla curvatura geodetica delle linee appartenenti ad una varietà qualunque*, RENDICONTI ACCADEMIA DEI LINCEI, vol. 31 (1922).

for the geodesic curvature κ, viz.,

$$\begin{aligned}\kappa=\sum_{rt}a_{rt}\left[\frac{d^2x_r}{ds^2}+\sum_{ik}\left\{\begin{matrix} i\ k \\ r\end{matrix}\right\}\frac{dx_i}{ds}\frac{dx_k}{ds}\right]&\\ \times\left[\frac{d^2x_t}{ds^2}+\sum_{ik}\left\{\begin{matrix} i\ k \\ t\end{matrix}\right\}\frac{dx_i}{ds}\frac{dx_k}{ds}\right]&,\end{aligned}\tag{4}$$

where $\left\{\begin{smallmatrix} i\ k \\ r\end{smallmatrix}\right\}$ is the well known Christoffel symbol of the second kind.

It is the purpose of this note to generalize the above procedure by replacing the geodesic g in (i) by any other curve $\bar{c}$ tangent to c at P, and (ii) by any other curve $\bar{c}$ having an infinitesimal element PQ in common with c.

2. *Generalization of the First Definition.* We have any two curves c and $\bar{c}$ tangent at P and two equal infinitesimal arc lengths PQ and $P\bar{Q}$ ($= \delta s$) on c and $\bar{c}$ respectively. Let the coordinates of P be x_r ($r = 1, 2, \cdots, n$), those of Q and $\bar{Q}$ (developing in powers of δs) be, respectively,

$$x_r + x_r'\delta s + \tfrac{1}{2}x_r''(\delta s)^2, \qquad (r = 1, 2, \cdots, n),$$
$$x_r + x_r'\delta s + \tfrac{1}{2}\bar{x}_r''(\delta s)^2, \qquad (r = 1, 2, \cdots, n),$$

(disregarding infinitesimals of higher order than the second), where the direction of the common tangent at P is given by $x_r' = dx_r/ds$ ($r = 1, 2, \cdots, n$), and where x_r'' and $\bar{x}_r''$ are the values of d^2x_r/ds^2 computed at P for c and $\bar{c}$ respectively. The differences of these coordinates are

$$\tfrac{1}{2}(x_r'' - \bar{x}_r'')(\delta s)^2,$$

and hence we have for the distance $Q\bar{Q}$ the expression

$$(Q\bar{Q})^2 = \sum (a_{rt})_Q(x_r'' - \bar{x}_r'')(x_t'' - \bar{x}_t'')(\delta s)^4/4,$$

where $(a_{rt})_Q$ represents the values of the coefficients a_{rt} at the point Q, i.e.,

$$(a_{rt})_Q = a_{rt} + \frac{da_{rt}}{ds}\delta s + \cdots. \tag{5}$$

Therefore we have

$$\left[\lim_{Q\to P}\frac{2Q\bar{Q}}{(\delta s)^2}\right]^2 = \sum_{rt} a_{rt}(x_r'' - \bar{x}_r'')(x_t'' - \bar{x}_t''). \tag{6}$$

3. *Generalization of the Second Definition.* We have any two curves c and $\bar{c}$ having an infinitesimal element $PQ = \delta s$ in common (i.e., as $Q \to P$, the curves c and $\bar{c}$ will approach tangency at P). The immediately following elements QR and $Q\bar{R}$ of c and $\bar{c}$ will form an infinitesimal angle $\delta\omega$ at Q. Let the direction PQ have for parameters $x_r' = dx_r/ds$ $(r = 1, 2, \cdots, n)$, i.e.,

$$\sum_{rt} a_{rt} x_r' x_t' = 1. \tag{7}$$

The directions of QR and $Q\bar{R}$ may be expressed by the parameters $x_r' + x_r''\delta s$, $x_r' + \bar{x}_r''\delta s$ (disregarding infinitesimals of higher order than the first), bound by the relations

$$\begin{cases} \sum_{rt} (a_{rt})_Q (x_r' + x_r''\delta s)(x_t' + x_t''\delta s) = 1, \\ \sum_{rt} (a_{rt})_Q (x_r' + \bar{x}_r''\delta s)(x_t' + \bar{x}_t''\delta s) = 1. \end{cases} \tag{8}$$

The angle $\delta\omega$ between these two directions at Q is given by

$$\cos \delta\omega = \sum_{rt} (a_{rt})_Q (x_r' + x_r''\delta s)(x_t' + \bar{x}_t''\delta s).$$

Using the first identity (8), this becomes

$$\cos \delta\omega = 1 + \sum_{rt} (a_{rt})_Q (x_r' + x_r''\delta s)(\bar{x}_t'' - x_t'')\delta s. \tag{9}$$

Subtracting the identities (8), we have

$$2\sum_{rt} (a_{rt})_Q (x_r' + x_r''\delta s)(\bar{x}_t'' - x_t'')\delta s + \sum_{rt} (a_{rt})_Q (\bar{x}_r'' - x_r'')(\bar{x}_t'' - x_t'')(\delta s)^2 = 0,$$

and introducing this into (9), we find

$$\cos \delta\omega = 1 - \tfrac{1}{2}\sum_{rt} (a_{rt})_Q (x_r'' - \bar{x}_r'')(x_t'' - \bar{x}_t'')(\delta s)^2. \tag{10}$$

On the other hand,

$$\cos \delta\omega = 1 - \tfrac{1}{2}(\delta\omega)^2 + \cdots. \tag{11}$$

Comparing (10) and (11), we deduce that

$$\lim_{Q \to P} \left(\frac{\delta\omega}{\delta s}\right)^2 = \sum_{rt} a_{rt}(x_r'' - \bar{x}_r'')(x_t'' - \bar{x}_t''). \tag{12}$$

We note that the right members of (6) and (12) are identical.

4. *Relative Curvature.* The expression

$$\sum_{rt} a_{rt}(x_r'' - \bar{x}_r'')(x_t'' - \bar{x}_t'') \tag{13}$$

has an interesting geometric interpretation. Let us set

$$A_r = \sum_{ik} \left\{ \begin{matrix} i\,k \\ r \end{matrix} \right\} x_r' x_k'. \tag{14}$$

We may then write (13) in the form

$$\begin{aligned} &\sum_{rt} a_{rt}[(x_r''+A_r)-(\bar{x}_r''+A_r)][(x_t''+A_t)-(\bar{x}_t''+A_t)] \\ (15) \quad &= \sum_{rt} a_{rt}(x_r''+A_r)(x_t''+A_t)+\sum_{rt} a_{rt}(\bar{x}_r''+A_r)(\bar{x}_t''+A_t) \\ &\qquad -2\sum_{rt} a_{rt}(x_r''+A_r)(\bar{x}_t''+A_t). \end{aligned}$$

We here introduce the geodesic curvature κ as given by (4), and the direction of the principal geodesic normal to a curve c at a point P as given by the parameters *

$$\mu^{(r)} = \frac{1}{\kappa}(x_r'' + A_r), \quad (r = 1, 2, \cdots, n), \tag{16}$$

so that

$$\sum_{rt} a_{rt}(x_r'' + A_r)(\bar{x}_t'' + A_t) = \kappa\cdot\bar{\kappa}\sum_{rt} a_{rt}\mu^{(r)}\bar{\mu}^{(t)} = \kappa\cdot\bar{\kappa}\cos\theta,$$

where θ is the angle between the principal geodesic normals to c and $\bar{c}$ at P. Then (15) or (13) takes the form

$$\kappa^2 + \bar{\kappa}^2 - 2\kappa\bar{\kappa}\cos\theta. \tag{17}$$

Finally, combining (6), (12) and (17), we have

$$\lim_{Q\to P} \frac{2Q\bar{Q}}{(\delta s)^2} = \lim_{Q\to P}\frac{\delta\omega}{\delta s} = \sqrt{\kappa^2 + \bar{\kappa}^2 - 2\kappa\bar{\kappa}\cos\theta}. \tag{18}$$

If, in the definitions (i) and (ii) of § 1, we replace the curve c and the tangent geodesic g by any two tangent curves c and $\bar{c}$, we shall say that the limiting expressions in (1) and (2) define the curvature of c relative to $\bar{c}$ or the *relative curvature* of c and $\bar{c}$. We may now state the following theorem.

Given any two curves c and $\bar{c}$ in V_n tangent at a point P. Let κ and $\bar{\kappa}$ be their respective geodesic curvatures and let θ be the angle between their principal geodesic normals at P. Then the relative curvature λ of c and $\bar{c}$ at P is given by

$$\lambda^2 = \kappa^2 + \bar{\kappa}^2 - 2\kappa\bar{\kappa}\cos\theta.$$

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* Bianchi, ibid., p. 364.

SQUARE-PARTITION CONGRUENCES *

BY E. T. BELL

1. *Introduction.* It is evident that the theory of partitions and that of the representation of an integer as a sum of squares must be closely interwoven since both originate in the elliptic theta and modular functions. In seeking the relations thus suggested, we find at the outset some remarkable types of congruences which deserve independent notice on account of their generality. Each congruence is to the odd prime modulus p; the most frequent type concerns the function expressing the number of ways in which an integer is a sum of p, $3p$, p^2, $3p^2$, p^s $(s > 0)$ or rp squares, where r is prime to p, and one of the following: the familiar denumerants of the classical theory of partitions; two new functions depending upon those partitions of an integer in which no part appears more than r times. Of the latter functions those corresponding to $r = 2, 3, 6$ play a central part in the entire theory. The subject is extensive. We shall give a sketch of the methods used sufficient for its systematic development. For the ϑ, q formulas see, e.g., Tannery-Molk, *Fonctions Elliptiques,* and note that we use Jacobi's theta notation (*Werke,* vol. 1, p. 501), so that π is omitted from ϑ_1'.

2. *Fundamental Identities.* In the usual notation $q_j = q_j(q)$,

$$(1)\qquad \begin{aligned} q_0 &= \Pi(1 - q^{2n}), & q_2 &= \Pi(1 + q^m), \\ q_1 &= \Pi(1 + q^{2n}), & q_3 &= \Pi(1 - q^m), \end{aligned}$$

extending to $n = 1, 2, 3, \cdots, m = 1, 3, 5, \cdots$, Euler's identities are

$$(2)\qquad q_1q_2q_3 = q_1(\sqrt{q})q_3 = 1, \qquad q_0 = \Sigma(-1)^n q^{3n^2+n},$$

Σ extending to $n = 0, \pm 1, \pm 2, \cdots$. Denote by $A_j(n, r)$ the coefficient † of q^{2n} in q_j^r $(j = 0, 1)$, of q^n in q_j^r $(j = 2, 3)$.

* Presented to the Society, April 7, 1923.

† The properties of these coefficients have been discussed and a practicable method for their numerical computation given in a paper which will be published in the American Journal.

By convention $A_j(0, r) = 1$. By (1) and the first of (2),

$$A_1(n, r) = (-1)^n A_2(n, -r) = A_3(n, -r) \qquad (r \gtreqless 0). \tag{3}$$

Henceforth p is an odd prime > 0, r an arbitrary integer > 0. If $s \neq 0, p$, the binomial coefficient $\binom{p}{s}$ is divisible by p, and hence by (1) we have, using Fermat's theorem,

$$q_j^{rp} \equiv q_j^r(q^p) \mod p, \qquad (j = 0, 1, 2, 3), \tag{4}$$

which means that the coefficients of like powers of the parameter q are congruent modulo p. Hence we have by (1) and (3), according as n is or is not prime to p,

$$A_j(n, rp) \equiv 0 \quad \text{or} \quad A_j(n/p, r) \mod p, \tag{5}$$

$$A_1(n, -rp) \equiv 0 \quad \text{or} \quad (-1)^n A_2(n/p, r) \mod p, \tag{6}$$

$$\begin{aligned}(-1)^n A_2(n, -rp) \equiv A_3(n, -rp) &\equiv 0 \\ &\text{or} \quad A_1(n/p, r) \mod p;\end{aligned} \tag{7}$$

and by (4) and the second of (2), according as n is not or is $\frac{1}{2}p(3a^2 + a)(a \gtreqless 0)$,

$$A_0(n, p) \equiv 0, \quad \text{or} \quad (-1)^a \mod p. \tag{8}$$

The summations referring to $n = 0, \pm 1, \pm 2, \cdots$, $m = \pm 1, \pm 3, \pm 5, \cdots$, we have $\vartheta_j = \vartheta_j(q)$,

$$\begin{gathered}\vartheta_0(-q) = \vartheta_3 = \Sigma q^{n^2}, \qquad \vartheta_1'(q^4) = \Sigma(-1|m)mq^{m^2}, \\ \vartheta_2(q^4) = \Sigma q^{m^2},\end{gathered} \tag{9}$$

where $(a|b)$ is the Legendre-Jacobi symbol, $(-1|a) = (-1)^{(a-1)/2}$ for a odd; and

$$\begin{gathered}\vartheta_3 = q_0 q_2^2, \qquad \vartheta_2(q^4) = 2q_0(q^4)q_1^2(q^4)q, \\ \vartheta_1'(q^4) = 2q_0^3(q^4)q.\end{gathered} \tag{10}$$

From the last of these, it follows by (4) that

$$A_0(n, 3p) \equiv 0 \quad \text{or} \quad (-1|a)a \mod p, \tag{11}$$

according as n is not or is $p(a^2 - 1)/8$ where $a > 0$ is odd.

The $A_j(n, r)$ are connected with partitions as follows. If in a given partition of n no part appears more than r times and each of precisely a_j distinct parts occurs exactly j times we call the hypercomplex number $(a_1, a_2, \cdots, a_r)$ the r index of the partition, and denote by $B_n(a_1, a_2, \cdots, a_r)$ or $B_n'(a_1, a_2, \cdots, a_r)$ the total number of partitions of n having this index according as all the parts are not or are restricted to be odd. As mentioned, the cases $r = 2, 3, 6$ are of special importance. For our purpose here it is sufficient by what precedes to consider in these cases $A_j(n, r)$ only when $j = 1, 2$. From (1) we have

$$A_1(n, 2) = \Sigma B_n(a_1, a_2)2^{a_1},$$
$$A_1(n, 3) = \Sigma B_n(a_1, a_2, a_3)3^{a_1+a_2},$$
$$A_1(n, 6) = \Sigma B_n(a_1, \cdots, a_6)2^{a_1+2a_3+a_5}3^{a_1+a_2+a_4+a_5}5^{a_2+a_3+a_4},$$

the summations referring to all $(a_1, a_2), \cdots, (a_1, \cdots, a_6)$ for n fixed.* The $A_2(n, r)$ for $r = 2, 3, 6$ are written down from these by accenting B.

Let $P(n)$, $Q(n)$, $R(n)$ denote respectively the total number of partitions of n, the number of partitions of n into odd parts, and the number of partitions of n into distinct odd parts. Then, from (1), (2), we have

$$A_0(n, -1) = P(n), \qquad A_1(n, 1) = Q(n),$$
$$A_2(n, 1) = (-1)^n A_3(n, 1) = R(n).$$

The square functions most frequently occurring are $N(n, r)$, the number of representations of n as a sum of r squares whose roots are $\gtreqless 0$, and $M(n, r)$, the number of representations of n as a sum of r odd squares whose roots are > 0. Obviously $M(n, r) = 0$ if n is not of the form $8k + r$, and

$$(12) \quad \vartheta_3{}^r(q) = \Sigma q^n N(n, r), \qquad \vartheta_2{}^r(q^4) = 2^r \Sigma q^{8n+r} M(8n + r, r),$$

where Σ refers to $n = 0, 1, 2, \cdots$, with the convention that $N(0, r) = 1$.

* The $A_j(n, 2)$ with a generalization have been fully discussed in a paper to appear in the ANNALS OF MATHEMATICS. They are remarkable as introducing for the first time a species of double periodicity into the theory of partitions. The $A_j(n, r)$, $r = 2, 3, 6, 9$, have been specially considered in the paper cited previously; they have many interesting connections with the class number for binary quadratic forms of a negative determinant.

Consider all the representations of n as a sum of r squares of numbers $\gtrless 0$ of the form $6k+1$, and let $S_0(n, r)$, $S_1(n, r)$ denote the respective numbers of these representations in which an even, an odd number of squares have square roots of the form $12k+7$. Write $S(n, r) = S_0(n, r) - S_1(n, r)$. Then $S(n, r)$ vanishes identically if n is not of the form $24k+r$, and from the second of (2) we have

$$(13) \qquad A_0(n, r) = S(24n + r, r),$$

whence it follows by (8) that

$$(14) \qquad S(24n + p, p) \equiv 0 \quad \text{or} \quad (-1)^a \mod p,$$

according as n is not or is $\frac{1}{2}p(3a^2 + a)(a \gtreqless 0)$, and by (11)

$$(15) \qquad S(24n + 3p, 3p) \equiv 0 \quad \text{or} \quad a(-1|a) \mod p,$$

according as n is not or is $p(a^2-1)/8$, where $a > 0$ is odd.

The congruences in this section appear to be sufficient for the systematic transposition of the classical theory of partitions into congruence relations of the type illustrated in the next. The labor of verifying the congruences numerically may be lightened by observing that $N(n, p)$ is congruent modulo p to twice the total number of representations of n as a sum of p squares with roots all > 0. Similar obvious remarks apply to any of the square functions encountered except those involving only odd squares.

3. *Congruences.* A short selection must suffice. Equating coefficients of like powers of q in $\vartheta_3^{rp} q_2^{-2rp} = q_0^{rp}$ we find

$$\Sigma A_2(s, -2rp) N(2n - s, rp) = A_0(n, rp),$$

the summation referring, as always henceforth unless otherwise noted, to all such $s \geqq 0$ as render the first arguments of the summands positive or zero. Applying (7) we get

$$(16) \quad \Sigma(-1)^s A_1(s, 2r) N(2n - sp, rp) \equiv A_0(n, rp) \mod p,$$

whence by (11) when $r = 3$,

$$(17) \quad \Sigma(-1)^s A_1(s, 6) N(2n - sp, 3p) \equiv 0 \quad \text{or} \quad a(-1|a) \mod p,$$

according as n is not or is $p(a^2 - 1)/8$, where $a > 0$ is odd. The ϑ, q identity being read in the alternative form $\vartheta_3^{rp} = q_0^{rp}q_2^{2rp}$ gives by (5) in the same way

$$(18) \qquad N(n, rp) \equiv \Sigma A_0(s, r)A_2(n - 2sp, 2rp) \mod p,$$

from which it follows by (5) that

$$(19) \qquad n \not\equiv 0 \mod p: \qquad N(n, rp) \equiv 0 \mod p.$$

From (18) and (5), we have

$$(20) \qquad N(np, rp) \equiv \Sigma A_0(s, r)A_2(n - 2s, 2r) \mod p,$$

and hence by (8),

$$(21) \quad N(np, p^2) \equiv \Sigma(-1)^a A_2(n - p(3a^2 + a), 2p) \mod p,$$

where Σ extends to all $a \gtreqless 0$ that make $n \geqq p(3a^2 + a)$. Applying (5) to (21) we have

$$(22) \qquad n \not\equiv 0 \mod p: \qquad N(np, p^2) \equiv 0 \mod p;$$

$$(23) \quad N(np^2, p^2) \equiv \Sigma(-1)^a A_2(n - (3a^2 + a), 2) \mod p,$$

where Σ extends to all $a \gtreqless 0$ that make $n \geqq 3a^2 + a$. Again from (20), (11) we find

$$(24) \quad N(np, 3p^2) \equiv \Sigma a(-1|a)A_2(n - \tfrac{1}{4}p(a^2-1), 6p) \mod p,$$

where Σ extends to all odd $a > 0$ that make $4n \geqq p(a^2 - 1)$.

Applying (5) to (24), we get

$$(25) \qquad n \not\equiv 0 \mod p: \qquad N(np, 3p^2) \equiv 0 \mod p,$$

$$(26) \quad N(np^2, 3p^2) \equiv \Sigma a(-1|a)A_2(n - \tfrac{1}{4}(a^2-1), 6) \mod p,$$

where Σ extends to all odd $a > 0$ that make $4n \geqq a^2 - 1$.

Similarly, from the second of (10), we find

$$(27) \qquad \Sigma A_1(s, -2r)M(8n - 8s + r, r) = A_0(n, r),$$

$$(28) \qquad M(8n + r, r) = \Sigma A_0(s, r)A_1(n - s, 2r).$$

To derive the associated congruences we replace r by rp

and proceed as before. Thus (27) gives

$$\Sigma(-1)^s A_2(s, 2r) M(8n + rp - 8sp, rp) \equiv A_0(n, rp) \mod p; \tag{29}$$

$$\Sigma(-1)^s A_2(s, 6) M(8n + 3p - 8sp, 3p) \equiv 0 \text{ or } a(-1|a) \mod p \tag{30}$$

according as n is not or is $p(a^2 - 1)/8$ where $a > 0$ is odd; while from (28),

(31) $M(8n + rp, rp) \equiv \Sigma A_0(s, r) A_1(n - sp, 2rp) \mod p;$

(32) $n \not\equiv 0 \mod p$: $M(8n + rp, rp) \equiv 0 \mod p;$

(33) $M(8np + rp, rp) \equiv \Sigma A_0(s, r) A_1(n - s, 2r) \mod p;$

(34) $M(8np+p^2, p^2) \equiv \Sigma(-1)^a A_1(n - \frac{1}{2}p(3a^2+a), 2p) \mod p,$

where Σ extends to all $a \gtreqless 0$ that make $2n \geqq p(3a^2 + a)$;

(35) $M(8np^2+p^2, p^2) \equiv \Sigma(-1)^a A_1(n - \frac{1}{2}(3a^2+a), 2) \mod p,$

where Σ extends to all $a \gtreqless 0$ that make $2n \geqq 3a^2 + a$;

(36) $n \not\equiv 0 \mod p$: $M(8np + p^2, p^2) \equiv 0 \mod p;$

$$M(8np + 3p^2, 3p^2) \equiv \Sigma a(-1|a) A_1(n - \tfrac{1}{8}p(a^2 - 1), 6p) \mod p; \tag{37}$$

(38) $n \not\equiv 0 \mod p$: $M(8np + 3p^2, 3p^2) \equiv 0 \mod p;$

$$M(8np^2 + 3p^2, 3p^2) \equiv \Sigma a(-1|a) A_1(n - \tfrac{1}{8}(a^2 - 1), 6) \mod p, \tag{39}$$

the summations in (37), (39) extending to all odd $a > 0$ making the first arguments of $A_1 \geqq 0$. Putting $r = 1$ in (29) and applying (13), (14) we find

$$\Sigma(-1)^s A_2(s, 2) M(8n + p - 8sp, p) \equiv 0 \text{ or } (-1)^a \mod p \tag{40}$$

according as n is not or is $\frac{1}{2}p(3a^2 + a)(a \gtreqless 0)$. Similarly, from (16), under the same conditions, we have

(41) $\Sigma(-1)^s A_1(s, 2) N(n - sp, p) \equiv 0$ or $(-1)^a \mod p.$

The number of congruences obtainable in this way is practically unlimited. Thus the memoir of Jacobi * on infinite series whose exponents are contained simultaneously in two different quadratic forms alone furnishes an inexhaustible supply, and the modular equations in elliptic functions give many more. Quadratic forms other than simple sums of squares appear in this connection. For example consider all the representations of n in the form

$$a_1^2 + a_2^2 + \cdots + a_r^2 + 3(b_1^2 + b_2^2 + \cdots + b_r^2),$$

in which $a_j,\ b_j \lesseqgtr 0$ and $a_j \equiv 1 \mod 6,\ b_j \equiv 1 \mod 4$ $(j = 1, 2, \cdots, r)$. Let $T_0(n, r)$ denote the total number of these representations in which an even number of the b_j are of the form $8k + 5$, and $T_1(n, r)$ the total number in which an odd number of the b_j are of the form $8k + 5$. Write

$$T(n, r) = T_0(n, r) - T_1(n, r).$$

Then Jacobi's result (*Werke*, vol. 2, p. 285)

$$q^4 q_0^2(q^{24}) = \Sigma(-1)^k q^{(6i+1)^2+3(4k+1)^2},$$

where Σ refers to $i, k = -\infty$ to $+\infty$, gives

$$A_0(n, 2r) = T(48n + 4r, r),$$

from which we find by successive applications of (5), for $s \geqq 0$,

(42) $n \equiv 0 \mod p$: $\quad T(48np^s + 4rp^{s+1}, rp^{s+1}) \equiv 0 \mod p,$

(43) $T(48np^{s+1} + 4rp^{s+1}, rp^{s+1}) \equiv A_0(n, 2r) \mod p,$

(44) $T(48np^{s+1} + 4p^{s+1}, p^{s+1}) \equiv A_0(n, 2) \mod p.$

Considerations of space preclude the giving of further examples.

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* *Werke*, vol. 2, pp. 219–288; CRELLE'S JOURNAL, vol. 37, pp. 61–94, 221–254.

DETERMINATION OF ALL SYSTEMS OF ∞^4 CURVES IN SPACE IN WHICH THE SUM OF THE ANGLES OF EVERY TRIANGLE IS TWO RIGHT ANGLES *

BY JESSE DOUGLAS

1. *Introduction.* Consider the curves which intersect an arbitrarily chosen system of ∞^1 curves in the plane under a fixed angle α. If α is varied, a system of ∞^2 curves is obtained, termed an *isogonal* family. Isogonal families are characterized by differential equations of the form

$$y'' = (T_x + y'T_y)(1 + y'^2), \tag{1}$$

where T is any function of x and y.

It is easy to prove synthetically that in all isogonal families, and in no other systems of ∞^2 curves in the plane, the sum of the angles of the triangle formed by any three of the curves is equal to π.†

A *natural* family of curves in any space is one obtainable as the system of extremals of a calculus of variations problem of the form

$$\int F ds = \text{minimum}, \tag{2}$$

where F is any point function.‡ In the plane, F is a function of x and y, and the Euler-Lagrange equation of (2) is

$$y'' = (L_y - y'L_x)(1 + y'^2), \tag{3}$$

where $L = \log F$.

Since the family formed by the ∞^2 straight lines of the plane is both isogonal and natural, and since each of these characters is invariant under conformal transformation, every

* Presented to the Society, April 28, 1923.

† G. Scheffers, *Isogonalkurven, Äquitangentialkurven und komplexe Zahlen*, Mathematische Annalen, vol. 60 (1905), p. 504.

‡ See E. Kasner, Princeton Colloquium Lectures (1912), pp. 34–37.

curve family conformally equivalent to the straight lines must be both isogonal and natural.

Conversely, for a curve family which is at once isogonal and natural, we have by identification of (1) with (3)

$$(4) \qquad T_x = L_y, \qquad T_y = -L_x.$$

These equations imply that $L - iT$ and consequently e^{L-iT} are analytic functions of $x + iy$. Since the conformal transformation

$$x_1 + iy_1 = e^{L-iT},$$

transforms $e^L ds$ into ds_1, it converts the extremals of $\int F ds = \int e^L ds =$ minimum into those of $\int ds_1 =$ minimum, that is, into the straight lines.

The conditions (4) can be satisfied only when T is Laplacian. It follows that the property of having the angle sum in each triangle equal to π is, in the plane, not restricted to the curve families derivable by conformal transformation from the straight lines. In contrast with this fact, it is the object of the present paper to prove the following theorem.

THEOREM. *If a system of ∞^4 curves in space is such that in each of its triangles the sum of the angles is equal to π, then it is either the system formed by the ∞^4 straight lines of space, or an image of that system by a conformal transformation of space, namely the ∞^4 circles through a fixed point.*

It is presumed in the statement of this theorem that the only systems of ∞^4 curves in space taken into consideration are those that can be defined by a system of two differential equations of the form

$$(5) \qquad y'' = F(x, y, z, p, q), \qquad z'' = G(x, y, z, p, q),$$

where p, q represent y', z' respectively, and where F and G are analytic functions of their five arguments. We select a region of the x, y, z, p, q continuum, within which each of these functions has a branch which is uniform and regular, and we restrict ourselves in the calculations that follow to this region and to these branches.

2. *A Triangle with One Infinitesimal Angle.* We shall use the symbol $\mathfrak{F}$ to denote any quadruply infinite family of curves defined by equations of the form (5). Under the restrictions just stated, there will pass through each point and in each direction a unique curve of $\mathfrak{F}$.

Choose any point 0—denote its coordinates by x_0, y_0, z_0—and let $1 : p_0 : q_0$ and $1 : p_0 + \delta p_0 : q_0 + \delta q_0$ define two directions through 0 infinitely near to one another. These determine in the family $\mathfrak{F}$ two consecutive curves C_1 and C_2. Let the equations of C_1 be

$$Y = y(X), \qquad Z = z(X); \tag{6}$$

then those of C_2 will be

$$\begin{aligned} Y &= y(X) + \delta p_0 \eta_1(X) + \delta q_0 \eta_2(X), \\ Z &= z(X) + \delta p_0 \zeta_1(X) + \delta q_0 \zeta_2(X), \end{aligned} \tag{7}$$

where η_1, ζ_1 and η_2, ζ_2 obey the equations of variation of the system (5)

$$\begin{aligned} \eta'' &= F_p\eta' + F_q\zeta' + F_y\eta + F_z\zeta, \\ \zeta'' &= G_p\eta' + G_q\zeta' + G_y\eta + G_z\zeta, \end{aligned} \tag{8}$$

and are completely determined by the additional data

$$(9_1) \quad \eta_1(x_0) = 0, \qquad \zeta_1(x_0) = 0, \quad \eta_1'(x_0) = 1, \qquad \zeta_1'(x_0) = 0,$$
$$(9_2) \quad \eta_2(x_0) = 0, \qquad \zeta_2(x_0) = 0, \quad \eta_2'(x_0) = 0, \qquad \zeta_2'(x_0) = 1.$$

It is to be understood that in the coefficients of (8), which are originally functions of x, y, z, p, q, we are to substitute by means of (6)

$$x = X, \quad y = y(X), \quad z = z(X), \quad p = y'(X), \quad q = z'(X),$$

so that these coefficients become functions only of X, the abscissa along C_1.

On C_1 let any point $1(x, y, z)$ other than 0 be selected, and let $2(x + \delta x, y + \delta y, z + \delta z)$ be an infinitely near point on C_2. Through 1 and 2 there passes a unique curve of the family $\mathfrak{F}$; designate it as $\bar{C}$, and denote by $1 : \bar{p} : \bar{q}$ and $1 : \bar{p} + \delta\bar{p} : \bar{q} + \delta\bar{q}$

its directions at 1 and 2 respectively. Then

$$\delta y = \bar{p}\delta x, \qquad \delta z = \bar{q}\delta x, \tag{10}$$

and

$$\delta\bar{p} = \overline{F}\delta x, \qquad \delta\bar{q} = \overline{G}\delta x, \tag{11}$$

where the bars over F and G indicate that these functions are to be formed for the arguments $x, y, z, \bar{p}, \bar{q}$.

Allow $1 : p : q$ and $1 : p + \delta p : q + \delta q$ to represent respectively the direction of C_1 at 1 and of C_2 at 2; then by differentiation of (6) and (7), substitution of the coordinates of 1 and 2, and use of (5),

$$p = y'(x), \qquad q = z'(x),$$

and

$$\left\{\begin{aligned} \delta p &= F\delta x + \delta p_0\eta_1' + \delta q_0\eta_2', \\ \delta q &= G\delta x + \delta p_0\zeta_1' + \delta q_0\zeta_2', \end{aligned}\right. \tag{12}$$

where the arguments in F, G are the x, y, z, p, q of the point 1 and curve C_1, and in $\eta_1', \zeta_1', \eta_2', \zeta_2'$ the argument is x.

The fact that the curve C_2 or (7) contains the point 2 gives, with the use of (10),

$$(\bar{p} - p)\delta x = \delta p_0\eta_1 + \delta q_0\eta_2,$$
$$(\bar{q} - q)\delta x = \delta p_0\zeta_1 + \delta q_0\zeta_2;$$

therefore

$$\left\{\begin{aligned} \delta x : \delta p_0 : \delta q_0 = \eta_1\zeta_2 - \eta_2\zeta_1 &: [(\bar{p} - p)\zeta_2 - (\bar{q} - q)\eta_2] \\ &: [-(\bar{p} - p)\zeta_1 + (\bar{q} - q)\eta_1]. \end{aligned}\right. \tag{13}$$

In the curvilinear triangle 012 or $C_1C_2\overline{C}$ let $\delta\omega$ denote the interior angle at 0, θ the interior angle at 1, and $\theta + \delta\theta$ the exterior angle at 2. Then the condition for an angle sum equal to two right angles is

$$\delta\theta = \delta\omega. \tag{14}$$

Now

$$\cos\theta = \frac{1 + p\bar{p} + q\bar{q}}{\sqrt{1 + p^2 + q^2}\sqrt{1 + \bar{p}^2 + \bar{q}^2}}.$$

Therefore

$$
(15)\qquad
\begin{aligned}
-\sin\theta\delta\theta = {} & \frac{1+p\bar p+q\bar q}{\sqrt{1+p^2+q^2}\sqrt{1+\bar p^2+\bar q^2}}\\
& \times\left\{\frac{\bar p\delta p+\bar q\delta q+p\delta\bar p+q\delta\bar q}{1+p\bar p+q\bar q}\right.\\
& \left.-\frac{p\delta p+q\delta q}{1+p^2+q^2}-\frac{\bar p\delta\bar p+\bar q\delta\bar q}{1+\bar p^2+\bar q^2}\right\},
\end{aligned}
$$

where the value of $\sin\theta$ is

$$
(16)\qquad \frac{\sqrt{(1+q^2)(\bar p-p)^2-2pq(\bar p-p)(\bar q-q)+(1+p^2)(\bar q-q)^2}}{\sqrt{1+p^2+q^2}\sqrt{1+\bar p^2+\bar q^2}}.
$$

Besides,

$$
(17)\qquad \delta\omega=\frac{\sqrt{(1+q_0^2)\delta p_0^2-2p_0q_0\delta p_0\delta q_0+(1+p_0^2)\delta q_0^2}}{1+p_0^2+q_0^2}.
$$

Combining the equations (14) to (17), we have

$$
(18)\qquad
\begin{aligned}
&-(1+p\bar p+q\bar q)\left\{\frac{\bar p\delta p+\bar q\delta q+p\delta\bar p+q\delta\bar q}{1+p\bar p+q\bar q}\right.\\
&\qquad\left.-\frac{p\delta p+q\delta q}{1+p^2+q^2}-\frac{\bar p\delta\bar p+\bar q\delta\bar q}{1+\bar p^2+\bar q^2}\right\}\\
&=\frac{1}{1+p_0^2+q_0^2}\sqrt{(1+q_0^2)\delta p_0^2-2p_0q_0\delta p_0\delta q_0+(1+p_0^2)\delta q_0^2}\\
&\quad\times\sqrt{(1+q^2)(\bar p-p)^2-2pq(\bar p-p)(\bar q-q)+(1+p^2)(\bar q-q)^2}.
\end{aligned}
$$

By means of the substitutions indicated by (11) and (12), the use of (13), the introduction of

$$
(19)\qquad
\begin{cases}
\phi=\dfrac{(1+q^2)F-pqG}{1+p^2+q^2},\\[2ex]
\psi=\dfrac{-pqF+(1+p^2)G}{1+p^2+q^2},
\end{cases}
$$

also of symbols ω_i to represent the two-rowed determinants in the matrix

$$
\begin{vmatrix}
\eta_1 & \zeta_1 & \eta_1' & \zeta_1'\\
\eta_2 & \zeta_2 & \eta_2' & \zeta_2'
\end{vmatrix}
$$

as follows,

$$\omega_1 = (\eta\zeta), \qquad \omega_2 = (\eta\eta'), \qquad \omega_3 = (\eta\zeta'),$$
$$\omega_4 = (\zeta\eta'), \qquad \omega_5 = (\zeta'\zeta), \qquad \omega_6 = (\eta'\zeta'),$$

and of

$$\begin{aligned}\Omega_1 &= (1 + q^2)\omega_4 + pq\omega_5,\\ \Omega_2 &= -(1 + q^2)\omega_2 + pq\omega_3 - pq\omega_4 - (1 + p^2)\omega_5,\\ \Omega_3 &= pq\omega_2 - (1 + p^2)\omega_3,\end{aligned}$$

finally of

$$\text{(20)}\quad \begin{cases} I = (1 + p_0^2)\eta_1^2 + 2p_0q_0\eta_1\eta_2 + (1 + q_0^2)\eta_2^2,\\ II = (1 + p_0^2)\eta_1\zeta_1 + p_0q_0(\eta_1\zeta_2 + \eta_2\zeta_1) + (1 + q_0^2)\eta_2\zeta_2,\\ III = (1 + p_0^2)\zeta_1^2 + 2p_0q_0\zeta_1\zeta_2 + (1 + q_0^2)\zeta_2^2,\end{cases}$$

the condition (18) reduces to

$$\begin{aligned}&(1 + p^2 + q^2)\omega_1\{(\bar{\phi} - \phi)(\bar{p} - p) + (\bar{\psi} - \psi)(\bar{q} - q)\}\\ &+ \Omega_1(\bar{p} - p)^2 + \Omega_2(\bar{p} - p)(\bar{q} - q) + \Omega_3(\bar{q} - q)^2\\ \text{(21)}\quad &= \frac{1+p^2+q^2}{1+p_0^2+q_0^2}\sqrt{III(\bar{p}-p)^2-2II(\bar{p}-p)(\bar{q}-q)+I(\bar{q}-q)^2}\\ &\times\sqrt{(1+q^2)(\bar{p}-p)^2-2pq(\bar{p}-p)(\bar{q}-q)+(1+p^2)(\bar{q}-q)^2}.\end{aligned}$$

It is to be observed that ω_1, Ω_1, Ω_2, Ω_3, and I, II, III are independent of $\bar{p}$, $\bar{q}$.

Since the right member of (21) and the second line of the left member are homogeneous functions of the second degree in $\bar{p} - p$, $\bar{q} - q$, the same must be true of

$$\text{(22)}\qquad (\bar{\phi} - \phi)(\bar{p} - p) + (\bar{\psi} - \psi)(\bar{q} - q).$$

By Taylor's theorem,

$$\begin{aligned}\bar{\phi}-\phi&=\phi_p(\bar{p}-p)+\phi_q(\bar{q}-q)\\ &\quad+\tfrac{1}{2}\{\phi_{pp}(\bar{p}-p)^2+2\phi_{pq}(\bar{p}-p)(\bar{q}-q)+\phi_{qq}(\bar{q}-q)^2\}+\cdots,\\ \bar{\psi}-\psi&=\psi_p(\bar{p}-p)+\psi_q(\bar{q}-q)\\ &\quad+\tfrac{1}{2}\{\psi_{pp}(\bar{p}-p)^2+2\psi_{pq}(\bar{p}-p)(\bar{q}-q)+\psi_{qq}(\bar{q}-q)^2\}+\cdots.\end{aligned}$$

Necessary conditions that (22) be homogeneous of the second degree in $\bar{p} - p$, $\bar{q} - q$ are seen to be

$$\phi_{pp} = 0, \quad 2\phi_{pq} + \psi_{pp} = 0, \quad \phi_{qq} + 2\psi_{pq} = 0, \quad \psi_{qq} = 0,$$

a system of partial differential equations whose solution is

$$\begin{aligned} \phi &= \beta q^2 - \gamma pq + \lambda p + \mu q + \nu, \\ \psi &= \gamma p^2 - \beta pq + \lambda' p + \mu' q + \nu'; \end{aligned} \tag{23}$$

and these forms of ϕ, ψ are seen to be also sufficient for the condition in question. β, γ, λ, etc. are functions only of x, y, z.

The left member of (21) now becomes a rational entire function of $\bar{p} - p$, $\bar{q} - q$; in order that the same be true of the right member we must have

$$\frac{I}{1+p^2} = \frac{II}{pq} = \frac{III}{1+q^2}. \tag{24}$$

By means of (8), (9), and (20), I, II, III can be expressed as power series in $t = x - x_0$; we find

$$\begin{aligned} I &= (1+p_0^2)t^2 + \{(1+p_0^2)F_p{}^0 + p_0q_0F_q{}^0\}t^3 + \cdots, \\ II &= p_0q_0t^2 + \{(1+p_0^2)G_p{}^0 + p_0q_0(F_p{}^0 + G_q{}^0) \\ &\qquad + (1+q_0^2)F_q{}^0\}t^3 + \cdots, \\ III &= (1+q_0^2)t^2 + \{p_0q_0G_p{}^0 + (1+q_0^2)G_q{}^0\}t^3 + \cdots. \end{aligned} \tag{25}$$

Furthermore,

$$\begin{aligned} 1+p^2 &= 1+p_0^2 + 2p_0F_0t + \cdots, \\ pq &= p_0q_0 + (p_0G_0 + q_0F_0)t + \cdots, \\ 1+q^2 &= 1+q_0^2 + 2q_0G_0t + \cdots. \end{aligned} \tag{26}$$

When, after dividing the members of (25) by the corresponding ones of (26), we equate coefficients of t^3, we obtain

$$\begin{aligned} &\frac{(1+p^2)F_p + pqF_q - 2pF}{1+p^2} \\ &= \frac{(1+p^2)G_p + pq(F_p+G_q) + (1+q^2)F_q - (pG+qF)}{2pq} \\ &= \frac{pqG_p + (1+q^2)G_q - 2qG}{1+q^2}, \end{aligned} \tag{27}$$

where the subscript and superscript 0 has been dropped with evident justification.

Writing

$$F = (1 + p^2)\phi + pq\psi, \qquad G = pq\phi + (1 + q^2)\psi, \tag{28}$$

obtained by reversion of (19), and then applying the values of ϕ and ψ given by (23), we find that (27) imposes the conditions

$$\nu = \beta, \qquad \nu' = \gamma, \qquad \lambda' = 0, \qquad \mu = 0, \qquad \lambda = \mu',$$

which by (23) and (28) reduce F and G to

$$F = (\beta - \alpha p)(1 + p^2 + q^2), \qquad G = (\gamma - \alpha q)(1 + p^2 + q^2), \tag{29}$$

where we have replaced $\lambda = \mu'$ by $-\alpha$.

3. *The* ∞^3 *Surfaces* Σ. Relative to the ∞^4 straight lines of space there is a family of ∞^3 surfaces, namely the planes, of which the straight lines are the mutual intersections. Not every system of ∞^4 curves in space has a so related family of ∞^3 surfaces. We have found the necessary and sufficient conditions to be

$$\begin{aligned} &F_q(F_p + G_q) + 4F_z - 2F_q' = 0, \\ &F_p^2 - G_q^2 + 4(F_y - G_z) - 2(F_p' - G_q') = 0, \\ &G_p(F_p + G_q) + 4G_y - 2G_p' = 0. \end{aligned} \tag{30}$$

The accent denotes the operator

$$\frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + F\frac{\partial}{\partial p} + G\frac{\partial}{\partial q}\,.$$

Let A be any point of space, through which d_1 and d_2 are any two directions, and let Γ_1 and Γ_2 denote the curves of the family $\mathfrak{F}$ which pass through A in these directions respectively. Choose arbitrarily a point B on Γ_1 and a point C on Γ_2. There is a unique curve BC of $\mathfrak{F}$ which passes through B and C.

On BC let D be any point. Suppose the curve determined by A and D to have at A the direction d.

Then under the hypothesis of an angle sum in every triangle equal to π, the direction d must belong to the flat pencil determined by d_1 and d_2. For by applying this hypothesis to the triangles ABD, ACD, ABC, it may be deduced that

$$\measuredangle d_1 d + \measuredangle d d_2 = \measuredangle d_1 d_2,$$

which implies that d is coplanar with d_1 and d_2.

The curves of $\mathfrak{F}$ which radiate from A in the directions of the pencil determined by d_1 and d_2 form a surface Σ. By the above, Σ contains AD; therefore it contains D; therefore it contains the curve BC, since D was an arbitrary point of BC.

Now B and C were arbitrary points on Γ_1 and Γ_2 respectively. It follows that through each pair of intersecting curves of $\mathfrak{F}$ there passes a surface Σ which carries ∞^2 curves of $\mathfrak{F}$. It is easy to see how this implies that the ∞^4 curves of $\mathfrak{F}$ are the mutual intersections of ∞^3 surfaces.

The functions F and G, so far reduced to the forms (29), must therefore obey (30). The imposition of (30) restricts F and G further, namely to the forms

$$\text{(31)} \qquad \begin{cases} F = (L_y - pL_x)(1 + p^2 + q^2), \\ G = (L_z - qL_x)(1 + p^2 + q^2), \end{cases}$$

where either

$$\text{(32}a\text{)} \qquad L = f(x^2 + y^2 + z^2),$$

or

$$\text{(32}b\text{)} \qquad L = g(y).$$

In the reduction of L, use is made, in general, of a transformation of the axes.

In other words, the differential equations of $\mathfrak{F}$ are now reduced to one or the other of the forms

$$\text{(33}a\text{)} \qquad \begin{aligned} y'' &= \rho(y - y'x)(1 + y'^2 + z'^2), \\ z'' &= \rho(z - z'x)(1 + y'^2 + z'^2), \qquad \rho = 2f'(x^2 + y^2 + z^2); \end{aligned}$$

$$\text{(33}b\text{)} \qquad \begin{aligned} y'' &= g'(y)(1 + y'^2 + z'^2), \\ z'' &= 0. \end{aligned}$$

4. *Final Conditions.* We consider first (33*a*). Its two equations can be combined so as to give

$$\begin{vmatrix} x & y & z \\ 1 & y' & z' \\ 0 & y'' & z'' \end{vmatrix} = 0,$$

whose general integral is $Ax + By + Cz = 0$, where A, B, C are arbitrary constants. It follows that each curve of $\mathfrak{F}$ lies in a plane that passes through the origin O. The ∞^2 curves of $\mathfrak{F}$ carried by each surface Σ are therefore the sections of Σ by the planes through O (provided that Σ is not itself a plane through O *).

We next observe that the equations (31) are the Euler-Lagrange equations for $\int e^L ds =$ minimum. But if a curve on a surface Σ is an extremal of $\int e^L ds$ relative to space, it is a fortiori an extremal relative to Σ. Thus the ∞^2 curves on Σ form, according to the definition in § 1, a *natural* family on Σ.

Imagine Σ to be represented conformally on the plane. Then this natural family on Σ goes over into a natural family in the plane. For conformal transformation multiplies ds by a point function; therefore the extremals of an integral of the form $\int$ *point function*$\cdot ds$ are transformed into the extremals of an integral of the same form.

Moreover, in this family of ∞^2 curves in the plane the sum of the angles of every triangle is equal to π; the family is therefore *isogonal* as well as natural. By § 1, it is therefore convertible into the ∞^2 straight lines by a conformal transformation of the plane. In this way, it is possible to convert the original system of ∞^2 curves on Σ into the straight lines of the plane by a conformal representation of Σ on the plane.

We have therefore to deal with the following problem: If O is a point of space, and Σ a surface not a plane through O, what is implied by the circumstance that Σ admits of a

* Each curve of $\mathfrak{F}$ belongs to ∞^1 surfaces Σ. These cannot all be planes through O unless the curve is a straight line through O, a case which may be laid aside without affecting the validity of our ultimate conclusion.

conformal representation on the plane in which its ∞^2 sections by the planes through O go over into the ∞^2 straight lines of the plane?

We say that Σ must be either a plane, or else a sphere passing through O.

For consider any minimal line of the plane. In the conformal representation it must correspond to a curve on Σ which (1) is a plane curve, (2) has zero length. A plane curve of zero length is necessarily a minimal straight line. The surface Σ therefore carries two systems of minimal straight lines.

A surface doubly covered by minimal straight lines is either a plane or a sphere. If Σ is a plane, the condition in question is satisfied without further restriction.

But if Σ is a sphere, it must, in addition, pass through O. For the angle sum in a triangle formed by three circles on a sphere, whose planes intersect in a point O, is greater than π if O is interior to the sphere, and less than π if O is exterior to the sphere. Moreover, the condition Σ is a sphere which passes through O, is a sufficient one, for then the stereographic projection of Σ from O as pole is a conformal representation of Σ on the plane which converts the sections of Σ by the planes through O into straight lines.

It follows that each curve of $\mathfrak{F}$ must be either a straight line, or a circle through O. Since each of the three curve families $\mathfrak{F}$, the straight lines of space, the circles through O, is an irreducible analytic manifold of four dimensions, $\mathfrak{F}$ must be identical either with the ∞^4 straight lines of space, or with the ∞^4 circles through O.

The case (33b) can by a similar argument be proved to lead only to the straight lines.

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VOLUME III OF LIE'S MEMOIRS

Sophus Lie's Gesammelte Abhandlungen (Samlede Avhandlinger). Edited by Friedrich Engel and Poul Heegaard. Volume III. *Abhandlungen zur Theorie der Differentialgleichungen*, erste Abteilung (Avhandlinger til Differentialligningernes Teori, første Avdeling), edited by Friedrich Engel. Leipzig, B. G. Teubner, and Kristiania, H. Aschehoug and Co., 1922. xvi + 789 pages.

Twenty-three years after the death of Sophus Lie appears the first volume to be printed of his collected memoirs. It is not that nothing has been done in the meantime towards making his work more readily available. A consideration of the matter was taken up soon after his death but dropped owing to the difficulties in the way of printing so large a collection as his memoirs will make. An early and unsuccessful effort to launch the enterprise was made by the officers of Videnskapsselskapet i Kristiania; but plans did not take a definite form till 1912; then through the Mathematisch-physische Klasse der Leipziger Akademie and the publishing firm of B. G. Teubner steps were taken to launch the project. Teubner presented a plan for raising money by subscription to cover a part of the cost of the work and a little later invitations to subscribe were sent out. The responses were at first not encouraging; from Norway, the homeland of Lie, only three subscriptions were obtained in response to the first invitations.

In these circumstances, Engel, who was pressing the undertaking, resorted to an unusual means. He asked the help of the daily press of Norway. On March 9, 1913, the newspaper TIDENS TEGN of Christiania carried a short article by Engel with the title *Sophus Lies samlede Afhandlinger* in which was emphasized the failure of Lie's homeland to respond with assistance in the work of printing his collected memoirs. This attracted the attention of the editor and he took up the campaign: two important results came from this, namely, a list of subscriptions from Norway to support the undertaking and an appropriation by the Storthing to assist in the work. By June the amount of support received and promised was sufficient to cause Teubner to announce that the work could be undertaken; and in November the memoirs for the first volume were sent to the printer, the notes and supplementary matter to be supplied later.

The Great War so interfered with the undertaking that it could not be continued, and by the close of the war circumstances were so altered that the work could not proceed on the basis of the original subscriptions and understandings and new means for continuing the work had to be sought. Up to this time the work had been under the charge of Engel as editor. But it now became apparent that the publication of the memoirs would have to become a Norwegian undertaking. Accordingly, Poul Heegaard became associated with Engel as an editor. The printing of the work became an enterprise not of the publishers but of the societies which support them in this undertaking. Under such circumstances the third volume of the series, but the first one to be printed, has now been put into our hands. "The printing of further volumes will be carried through gradually as the necessary means are procured; more I cannot say about it," says Engel, "because the cost of printing continues to mount incessantly."

The general plan contemplates the publication of Lie's memoirs in six volumes while a seventh volume is to be devoted to the principal works found among Lie's literary remains. Volumes I and II are to contain the memoirs on geometry; volumes III and IV, those on differential equations; and volumes V and VI, those on transformation groups. Naturally these three divisions are not rigorously separated one from another; in fact, in most of the memoirs in a given part appears matter relating also to the other two parts. In each part the memoirs are arranged chronologically; for the most part, the material in the first of two related volumes is that which was first published at Christiania, while the material in the second is taken principally from the MATHEMATISCHE ANNALEN and the publications of the Leipzig Akademie. In this way an arrangement is effected by which two articles never appear in the same volume one of which is a reworking of the other. Every memoir already has its definite place assigned to it in the completed work and a definite number in the volume to which it belongs is given to it. In this way it becomes possible to give cross references from volume to volume without any possibility of confusion. In the (unexpected) event of the discovery of an article at first overlooked, it will be assigned to its chronological position, with a number followed by a star, so as not to disturb the numbering of the other articles.

To facilitate the examination of references by means of the collected works and without the use of the articles in their original place of publication, the exact paging of the original articles is indicated by means of numbers inserted in the lines at the appropriate places. This is a feature of convenience which is to be commended. By means of it the paging of the original is easily determined from the collected memoirs.

The publication is started with the third volume because the letters of Lie furnish such an abundant material for the explanation of the memoirs in it; and this material has been used freely. The letters to A. Mayer have been particularly useful in this respect. Besides the material gathered from the letters there is a rich copiousness of notes and explanations by Engel. These have been prepared with the purpose of making the memoirs in this volume more readily understood by the reader. The editor anticipates that the reader may sometimes find these too extensive; but, in case this evil is found to exist in them, he has thought that it is much less than the possible opposite evil of notes which are not sufficiently full. The theory of partial differential equations of the first order and of Pfaff's problem was greatly modified by the researches of Lie. Concerning the elucidation of these researches in the notes Engel speaks as follows in his preface: "Ich habe nun versucht auseinanderzusetzen, wie Lie etwa ursprünglich zu seinen Sätzen gelangt ist, namentlich habe ich mich bemüht, für die Theorie der partiellen Differentialgleichungen 1.O. nach Möglichkeit den Standpunkt wiederherzustellen, auf dem Lie war, als er die ersten Abhandlungen des Bandes (Nr. I–IV) veröffentlichte. Obwohl ich mir bewusst war, mir damit eine äusserst schwierige, fast unlösbare Aufgabe gestellt zu haben, habe ich doch den Versuch gewagt, und ich hoffe, in den Anmerkungen zu Abhandlung I [covering 38 pages] wenigstens zum Teil das Richtige getroffen zu haben; vollständig befriedigt bin ich auch selber noch nicht, obwohl ich gerade diese Anmerkungen ein halbesdutzendmal umgearbeit habe. Sicherer bin ich meiner Sache beim Pfaffschen Probleme; ich darf wohl behaupten, dass meine Anmerkungen zu Abhandlung XI [16 pages] den ursprünglichen Gedankengang Lies im wesentlichen richtig wiedergeben."

The memoirs of Lie reprinted in the present volume cover 562 of the total 805 pages. But the notes and supplementary matter are printed much more compactly than the memoirs themselves, so that the former make up distinctly more than one third of the whole volume. This additional material has been prepared with great care and with the convenience of the reader always in mind. The articles originally printed in Norwegian appear now in a German translation. The few notes in French are printed in that language except in the case of notes which Lie himself had published both in French and in German, these latter being reprinted in German with a reference also to the place of publication in French. All of Lie's earlier work placed by the editors under differential equations is therefore readily available to all who read German (the material in the French notes being only brief abstracts); and this is placed before the reader in convenient form with an ample richness of explanatory material.

In his preface Engel insists upon certain qualities of Lie's work—qualities which partly explain the copiousness of the notes which he has inserted in the present volume. The ideas and points of view which Lie possessed and never adequately made known to others were irretrievably lost with his death. In his printed works there are many suggestions which were not followed up by him. These are scattered, and are largely inaccessible. When his complete works are before us in convenient form, these fragmentary suggestions can be seen in their totality and they will throw light one upon another. In this way Lie will become his own interpreter. This will be particularly true when the material in the letters is brought to bear upon the researches to which its parts refer. It seems important therefore to bring the scattered suggestions of Lie together in connection with the memoirs to which they may properly be attached.

The editor's careful work in preparing this volume for the printer and in seeing it through the press has evidently been for him a labor of love; in performing it he has rendered a great service to all those who will have occasion to use the memoirs of Lie which are reprinted in this volume. At present the intention is that the fifth volume shall follow the third; but no definite promises are being made owing to the difficulties which we have already named. The memoirs for the fifth volume are now ready for the printer; but the notes for it will be prepared while the volume is in course of being printed. It is impossible yet to tell what the extent of these notes will be; but the editor states that they will be far less extensive than those in the present volume. All mathematical libraries will of course procure copies of the volume now printed. It is to be hoped also that every individual who can utilize a private copy of the volume effectively will purchase one so as to facilitate the publication of the next volumes.

R. D. CARMICHAEL

BOLZANO ON PARADOXES

Paradoxien des Unendlichen. By Bernard Bolzano, mit Anmerkungen versehen von Hans Hahn. Leipzig, Meiner, 1921. 12 + 157 pp.

This is a new edition, with notes by Hans Hahn, of the book first published in 1851, three years after Bolzano's death. It appears to have remained for many years almost unknown; for L. Couturat makes no mention of it in his book *De L'Infini Mathématique* (1896) except to state in a note that he came across it when his own book had already been entirely printed. Stolz includes a consideration of this book in an article (MATHEMATISCHE ANNALEN, 1881, p. 255) giving an estimate of Bolzano's work and influence, and states that several years before Cauchy published his lectures on the calculus, Bolzano had developed the fundamental concepts of the calculus which in many respects agree with those of Cauchy, but which in important respects are more complete. Furthermore Hankel attributes to Bolzano priority over Cauchy of the proper conception of infinite series.

The reviewer does not think it fair to criticize severely from the standpoint of present standards of rigor a book written about seventy-five years ago; but in view of the high estimates put on Bolzano's work, it does seem desirable to mention a few of the errors.

Bolzano is concerned with a discussion of Gergonne's solution of the series

$$(1) \qquad a - a + a - a + \cdots.$$

The solution is as follows. Let x be the value of the series; then

$$x = a - a + a - a + \cdots = a - (a - a + a - a + \cdots),$$

that is,

$$x = a - x, \qquad \text{and} \qquad x = \frac{a}{2}.$$

Bolzano criticizes this solution in two ways. In the first place he says that the series in parentheses is not identical with the series (1), because regarded as a set of terms it lacks the first term a. This assertion of Bolzano's is of course erroneous; in fact if it be granted that the series (1) have a value at all, then Gergonne's solution is correct. In the second place Bolzano objects altogether to attaching a value to the series (1). While this objection is quite legitimate, the *grounds* for the objection are not. The series can have no value, Bolzano says, since if it did have, it would simultaneously be equal to 0, a and $-a$, inasmuch as it can be written in several forms, as follows:

$$\begin{aligned} a - a + a - a + \cdots & \\ &= (a - a) + (a - a) + \cdots \\ &= a + (-a + a) + (-a + a) + \cdots \\ &= (-a + a) + (-a + a) + \cdots \\ &= -a + (a - a) + (a - a) + \cdots. \end{aligned}$$

The author's demand, however, that a satisfactory definition for the value of a series must allow for the interchange of the order of terms and for the insertion and omission of parentheses, is of course an unreasonable one; it is not indeed satisfied even for all convergent series.

Bolzano now discusses the two series

$$S_1 = 1 + 2 + 3 + 4 + \cdots,$$
$$S_2 = 1 + 2^2 + 3^2 + 4^2 + \cdots.$$

It is pointed out, on the one hand, that $S_1 < S_2$, inasmuch as each term of S_1 (except the first) is less than the corresponding term of S_2; but on the other hand, $S_1 > S_2$, since every term of S_2 occurs somewhere in S_1 and there are terms in S_1 which do not occur in S_2. Bolzano's explanation of this paradox is that in reality $S_2 > S_1$; for one has no right to infer that the first of two infinite series has a greater value than the second simply on the ground that the first contains all the terms of the second and many other terms besides. This explanation is of course entirely incorrect. The fact is that neither of the two series can be asserted to be greater in value than the other, because as neither of them possesses a value, their values cannot be compared.

There are misstatements in the theory of functions. The author states (p. 65, footnote) that every continuous function, except possibly for isolated values of the independent variable, possesses a derivative, and (p. 68) can be expanded by Taylor's Theorem.

On the other hand Bolzano must be credited with ideas that were certainly in advance of his time. For example, he gives a proof (p. 13) of the existence of an infinite set—by proving that the number of propositions is infinite—and this anticipates Dedekind by nearly forty years. Again, he points out the possibility of setting up a one-one correspondence between the elements of an infinite set and those of a proper part of that set—an idea that was to become fundamental in the work of Cantor and Dedekind.

The book deals with paradoxes of the infinite. Some of these so-called paradoxes are, from the standpoint of present knowledge, not paradoxes at all. Such, for example, are those involving infinite series, mentioned above. Others are cleared up by an accurate application of the definitions of infinity and infinitesimal.

The book is of interest to the philosopher and to the theologian as well as to the mathematician. Bolzano was an Austrian priest, who as Professor of the Philosophy of Religion at Prague tried in his lectures to reconcile Catholic theology with the ideas of modern science. Twenty-five pages of the book are devoted to purely metaphysical considerations, which include a development of the doctrine of panpsychism and an application of this doctrine to the problem of interaction between mind and matter.

The notes by Hans Hahn constitute a helpful guide to the text. Explanations are given only to the mathematical portions of the text (pp. 1–107); the twenty-five pages of metaphysical discussion are not considered by Hahn.

L. L. Silverman

SHOTER NOTICES

Die Theorie der Gruppen von endlicher Ordnung, mit Anwendungen auf algebraische Zahlen und Gleichungen sowie auf die Kristallographie. By Andreas Speiser. Berlin, Julius Springer, 1923. viii + 194 pp.

This book is volume 5 of the series entitled *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete*, of which the first volume appeared in 1921. In accord with the plan of the series special attention is given to applications, but the greater part of the work is devoted to the development of the fundamental theorems relating to the theory of the groups of finite order. The beginner who may find the developments too brief is referred to the second volume of Weber's *Algebra* and to Netto's *Gruppen und Substitutionentheorie* for more extensive treatments.

From the title of the series it is clear that the aim is to deal with fundamental theories rather than to present details. Differences of opinion naturally exist as regards what should be regarded as most fundamental. Some might regard the theory of the ϕ-subgroups, which does not appear in the present volume, as more fundamental than some of the theories which do appear. In fact, some readers may not agree with a statement found on page 97 to the effect that the representation of groups by means of substitutions is the most important domain of group theory. Most readers will probably agree, however, that the material of the present volume has been, on the whole, wisely selected for the purposes in view, which seem to have included an introduction to the theories due to Frobenius. From this standpoint the present volume is especially useful.

The following list of the fifteen chapter headings may serve to exhibit the nature of the material treated: The foundations, invariant subgroups and factor-groups, abelian groups, conjugate subgroups, Sylow groups and p-groups, groups of crystallography, permutation groups, automorphisms, monomial groups, representation of groups by linear homogeneous substitutions, group-characteristics, applications of the theory of group-characteristics, arithmetical researches on substitution groups, groups of a given degree, and theory of equations.

Comparatively few errors seem to have escaped the notice of the author. Among the few that might possibly trouble the beginner are the following: On page 66 it is stated, in theorem 66, that the subgroups composed separately of all the substitutions of a transitive permutation group of degree n which omit a given letter constitute a system of n conjugate subgroups of index n. This is true only when these subgroups are actually of degree $n - 1$. On page 76 it is stated that the cyclic invariant subgroup of index 2 of a dihedral group gives rise to outer isomorphisms. This is evidently not the case when the order of this dihedral group is 6, 8, or 12. The first of these groups has no outer isomorphisms at all.

G. A. Miller

Repertorium der Höheren Mathematik. E. Pascal. Second edition; second volume, *Geometry*; second section, *Geometry of Space*. Edited by H. E. Timerding. Leipzig, B. G. Teubner, 1922. xii + 628 pp.

The first section of the volume on analysis and also of the one on geometry of this second edition of Pascal's *Repertorium of Higher Mathematics* appeared in 1910. They were reviewed in this BULLETIN (volume 19, pp. 372–374). The second section of the volume on geometry, after being on the eve of publication for nearly ten years, has now appeared.

This section is devoted to geometry of space. In a subject so extensive, the first problem is a proper selection of topics for consideration. Somewhat more than half of the work is devoted to algebraic surfaces. Of the remainder, about one third is devoted to algebraic curves, a third to differential geometry and the remainder to line geometry and algebraic transformations of space. It is to be regretted that some of the more recondite portions of these subjects were not omitted in favor of the elements of synthetic, particularly of projective, geometry of space.

The treatment of the various topics is always good and occasionally excellent. In so brief and comprehensive a volume, the treatment of any subject must devote itself mainly to fundamentals and be of interest chiefly to those who are not specialists in that subject. The needs of such readers are accordingly borne constantly in mind. Occasionally, the limitations of a theorem are not adequately stated, but in some of these cases the error is due to the carelessness of the investigator to whom the theorem is due. Occasionally also, technical terms are used which are neither explained in the context nor to be found in the index.

The authors have given in clear and compact form the principal facts of the topics discussed. Mathematicians in every field will find this a useful work for reference.

C. H. SISAM

Tables Logarithmiques à treize décimales. By H. Andoyer. Paris, J. Hermann, 1922. x + 26 pp.

This pamphlet gives the logarithms of numbers from 100 to 1000 and of numbers from 100,000 to 101,000 to thirteen places besides some shorter tables of first and second differences. The tables are designed, of course, for use in work in which the usual tables to seven or eight places would not give the desired accuracy. A mechanical computer would be a great aid in the use of the tables.

In using the tables every number N is first resolved into two factors n and N' where, in general, n is taken to be the first three digits of N but in any case the three digits so that the first three digits of N' shall be 100. It remains then merely to add to the logarithm of n taken from the first table the logarithm of N' taken from the second table. The table of corrections corresponding to the first and second differences is used, of course, in obtaining the logarithm of N'. The inverse problem of finding the antilogarithm of a logarithm involves no additional principles.

C. H. FORSYTH

Fundamental Congruence Solutions. By Lt.-Col. Allan Cunningham. London, Francis Hodgson, 1923. xviii + 92 pp.

Haupt-exponents, Residue-indices, Primitive Roots, and Standard Congruences. By Lt.-Col. Allan Cunningham. Francis Hodgson, 1922. viii + 136 pp.

The first of these two books gives in 92 pages one value of the root y, of the "fundamental congruence":

$$y^{\xi} \equiv +1 \pmod{p \text{ or } p^k}$$

for values of the modulus not exceeding 10,201. ξ is the "Haupt-exponent," or the exponent to which y belongs. Owing to the immense amount of labor necessary to determine it, the smallest value of y is not always the one given. If, however, the smallest value is not greater than 13, it is given. Also when $\xi = p - 1$ (and y is therefore a primitive root), the smallest value of y is listed. There are further cases, when ξ has certain other values, where the root y is the smallest root.

One root, y, being listed, the others may be determined. Rules are given to effect this. Some account of the method of computation of the tables appears in the introduction, together with a description of the checks employed.

The tables are arranged in two lines for each modulus, thus:

p	ξ, ν
	y

where ν is the complementary factor of ξ with respect to $p - 1$, so that $\xi\nu = p - 1$. The values of the modulus are arranged consecutively so far as the primes are concerned, but for some reason, not explained, and not quite clear to the reviewer, those moduli which are powers of primes are inserted, part on the eighth page and part on the ninetieth page. These lists are not hard to find when one knows where they are, but a note telling where they are hidden would save users of the table some little annoyance and bewilderment.

In 31 pages of the second book we are given for each prime p not greater than 10,000 the factorization of $p - 1$. Then is given also the exponent to which each of the numbers 2, 3, 5, 6, 7, 10, 11, 12 belongs. A primitive root for each prime is also indicated. Then comes a similar table for the powers of 2, followed by a page giving the same information for powers of odd primes. Pages 33 to 34 give the solutions (x_0, α_0, x_0') of the two congruences

$$\left.\begin{aligned} 2^{x_0} &\equiv \pm\, y^{\alpha_0} \\ 2^{x_0'}y^{\alpha_0} &\equiv \pm 1 \end{aligned}\right\} \pmod{p \text{ or } p^k}$$

for $y = 3, 5, 7, 11$, and p as before. A similar table follows in which the congruences are

$$\left.\begin{aligned} 10^{x_0} &\equiv \pm\, y^{\alpha_0} \\ 10^{x_0'}y^{\alpha_0} &\equiv \pm 1 \end{aligned}\right\} \pmod{p \text{ or } p^k}.$$

The final table of the book, beginning at page 97, is a continuation of the first, slightly abridged to accommodate the larger numbers that appear as p runs from 10,007 to 25,409 inclusive.

The value of a piece of tabular work may appear in either or both of two ways. The table may be of use in other computations; or it may serve to put complicated results before the eye in such a simple form as to reveal hidden relations. The first of these purposes is evidently the one intended for the two books of tables published by Lt.-Col. Cunningham. They are to serve as aids in the factorization of numbers of the form

$$y^n \pm 1.$$

A detailed explanation of their use in this connection, together with illustrative examples, would do much to increase their usefulness and availability to other workers in this field. The need for an extensive table of primitive roots is readily appreciated by any one working in the theory of numbers. If one should by chance encounter a congruence such as

$$y^{256} \equiv 1 \pmod{4297}$$

he would turn to these tables with gratitude, but just how a congruence of this sort might arise in connection with other parts of the theory of numbers is a question to which at least a few words might well be given.

D. N. LEHMER

Vector Calculus. By James Byrnie Shaw. New York, D. Van Nostrand Company, 1922. iv + 314 pp.

In his preface the author says that "he has examined the various methods that go under the name of vector, and finds that for all purposes of the physicist and for most of those of the geometer, the use of quaternions is by far the simplest in theory and in practice." This indicates clearly the point of view of the book. The quaternion notation is used but tables of other equivalent notations are given.

The first chapter is a historical sketch of the various systems of vector analysis. The next six chapters are concerned with scalar and vector fields, the algebraic combinations of vectors and the differential operations. These are illustrated by a large number of quantities occurring in geometry, electricity and magnetism, mechanics, theory of elasticity, etc., each of which is defined when introduced. The following two chapters give a systematic exposition of the differential and integral calculus of vectors, with applications to geometry and such topics as Laplace's equation, Green's theorem, and spherical harmonics. The remaining three chapters treat the linear vector function with applications to deformable bodies and hydrodynamics. Extensive lists of problems are given covering almost all the topics discussed.

The book impresses one as containing an extraordinary number of topics treated in a way that (to one acquainted with those topics) is interesting and easy to follow. Students of the better class will certainly acquire a considerable knowledge of mathematics and physics by studying this book. Whether an individual teacher chooses it, however, will probably depend on whether he is willing to use the quaternion notation or translate it into the form that he does use.

H. B. PHILLIPS

Tables of Applied Mathematics in Finance, Insurance, Statistics. By James W. Glover. Ann Arbor, Mich., George Wahr, 1923. xiii + 676 pp.

This book of tables is designed for use in connection with college textbooks on finance, insurance and statistics. The tables are quite complete and extensive for use in finance and insurance and would be very useful in many statistical investigations. The tables are divided into four parts, of which the last part, comprising almost 200 pages, is a photographic reproduction of Bruhn's table of logarithms to seven places.

Part I contains tables of compound interest functions (to eight places) and their logarithms (to seven places) for thirty-two rates of interest, those up to and including $1\frac{1}{2}$ per cent. for periods up to 200 and for higher rates up to 100. Some of the tables are published here for the first time. The tables include practically every conceivable period and rate of interest which would be involved in converting interest. There are also auxiliary tables to be used when interest is converted very frequently and even instantaneously. Tables of the sinking fund function are also included so that all of the six fundamental functions are tabulated. Logarithms of the factor $1 + i$ are also given to twenty places for certain rates of interest for computing very unusual values of functions.

Part II contains values of life insurance and disability functions, including ordinary commutation columns based on the American Experience table and at three rates of interest. Tables of functions to be used in computing terminal reserves, paid-up and extended insurance values, etc., are given for the rate 4 per cent. which are not included in Dawson's *Derived Tables*. Hunter's Makehamized table and commutation columns for two lives for work with joint lives, and Hunter's disability tables are useful divisions of this part.

Part III contains tables to be used in statistical work, some of which are excellent and very desirable. The most important are the tables of ordinates, areas and derivatives (second to eighth) of the normal curve, the logarithms of the Gamma function and squares, cubes, square and cube roots (of numbers up to 1000). One of the criticisms of the reviewer would be of the selections of some of the other tables given in this part. However, the latter tables occupy a relatively small space and it would be difficult if not impossible to select and include a set of tables for statistical work which would suit everyone.

Another criticism which applies to the whole set of tables, but particularly to this part, is of the lack of explanation of the functions tabulated. A short reference in the preface to a particular use to be made of certain ones of the tables is, with about one exception, about all the explanation offered. A short explanation and an application of each table given in this part (similar to those given in the preface of Pearson's *Tables for Statisticians*, etc.), given preferably on the page just preceding the table, would add much to the usefulness of the book. As a single example, the use of the page of logarithms of Bernoulli's numbers from $n = 1$ to $n = 200$ will probably be very much affected by the lack of an explanation, and some who have used Bernoulli's numbers before may have to make a preliminary and independent investigation to make sure of the notation used here.

The only other criticism is in regard to the omission in several cases of all signs of the negative part of a characteristic of a logarithm, especially in a table or among tables where a positive characteristic of more than 9 is not unexpected. The scheme followed in printing the tables no doubt had most to do with these omissions, but nevertheless the omissions are unfortunate.

The tables as a whole are excellent both in intrinsic value and typography, and are bound to be adopted widely as a standard in work for which they were designed. The feature which will probably commend them most is their completeness for work in finance with logarithms. Logarithms are a practical necessity in such work and a four-place table is in most cases scarcely less absurd than no table at all. Yet there is little doubt but that few of the many attempts made throughout this country to give courses in finance leave a satisfactory impression of the use of logarithms in such connections upon the mind of the student. There should be no further excuse for this kind of a situation with this book available.

The author's reputation for carefulness and reliability is enough to warrant the reliability of the tables. The author states in the preface that it is scarcely possible to compile so large a set of tables without a few errors creeping in somewhere.

C. H. Forsyth

Iamblichvs Theologovmena Arithmeticæ, edidit Victorivs de Falco. Lipsiæ in Ædibvs, B. G. Tevbneri, MCMXXII. xvii + 90 pp.

This booklet is a new edition of the Greek text of the *Theologovmena arithmeticæ* which is attributed by some critics to Iamblichus. It is edited after a careful re-examination of the various extant manuscripts and is accompanied by notes written in Latin. This publication will be of interest to students desiring to enter more intimately into the study of the relations of mathematics to philosophy and mysticism than is usually done in our histories of mathematics. It deals with the theologic aspect of numbers and their mystic relations to the various heathen deities, and with obscure cosmological speculations.

Florian Cajori

Vektoranalysis. By Siegfried Valentiner. Third edition. (Sammlung Göschen 354.) Berlin, Vereinigung Wissenschaftlicher Verleger, 1923. 132 pp.

This little manual is the slightly modified, rewritten edition of the original. The chief changes are in the use of smaller type and closer setting which has apparently reduced the amount. The only serious cut however is in the omission of the useful collection of formulas at the end of the preceding edition. Slight changes of the text occur, but none of importance.

J. B. Shaw

NOTES

At the meeting of the Southwestern Section of this Society, at the University of Missouri on December 1, Professor Henry Blumberg, of the University of Illinois will give an address on *Properties of unrestricted functions*, at the invitation of the program committee.

Three numbers of the TRANSACTIONS OF THIS SOCIETY have recently been published almost simultaneously, from three different presses. These numbers contain the following papers. Volume 23, number 4 (June, 1922): *A proof and extension of the Jordan-Brouwer separation theorem*, by J. W. Alexander; *Oscillation theorems in the complex domain*, by E. Hille; *On certain relations between the projective theory of surfaces and the projective theory of congruences*, by F. E. Wood; *Asymptotic planetoids*, by D. Buchanan; vol. 24, number 1 (July, 1922): *Associated sets of points*, by A. B. Coble; *On algebraic functions which can be expressed in terms of radicals*, by J. F. Ritt; *On the location of the roots of the jacobian of two binary forms, and of the derivative of a rational function*, by J. L. Walsh; *I-conjugate operators of an abelian group*, by G. A. Miller; volume 25, number 1 (January, 1923): *The* (1, 2) *correspondence associated with the cubic space involution of order two*, by F. R. Sharpe and V. Snyder; *Sur certaines équations aux différences finis*, by N. E. Nörlund; *Differential geometry of an m-dimensional manifold in a euclidean space of n dimensions*, by C. E. Wilder; *Expansions in terms of solutions of partial differential equations. First paper: Multiple Fourier series expansions*, by C. C. Camp; *Euler algebra*, by E. T. Bell.

The opening number of volume 45 of the AMERICAN JOURNAL OF MATHEMATICS (January, 1923) contains: *On the number of solutions in positive integers of the equation* $yz + zx + xy = n$, by L. J. Mordell; *A closed set of normal orthogonal functions*, by J. L. Walsh; *Congruences determined by a given surface*, by Claribel Kendall; *Linear partial differential equations with a continuous infinitude of variables*, by I. A. Barnett; *On the ordering of the terms of polars and transvectants of binary forms*, by L. Isserlis.

The concluding number of volume 23, series 2, of the ANNALS OF MATHEMATICS (June, 1922) contains: *On the positions of the imaginary points of inflexion and critic centers of a real cubic*, by B. M. Turner; *Frequency distributions obtained by certain transformations of normally distributed variates*, by H. L. Rietz; *The associated point of seven points in space*, by H. S. White; *Common solutions of two simultaneous Pell equations*, by A. Arwin; *On the complete independence of Hurwitz's postulates for abelian groups and fields*, by B. A. Bernstein; *On power series with positive real part in the unit circle*, by T. H. Gronwall; *Algebraic surfaces, their cycles and integrals. A correction*, by S. Lefschetz.

The following university courses in mathematics are announced for the academic year 1923–1924, in addition to those listed in the July number of this BULLETIN:

UNIVERSITY OF WISCONSIN (First Semester).—By Professor E. B. Skinner: Advanced calculus; Theory of numbers.—By Professor H. W. March: Harmonic analysis.—By Professor E. B. Van Vleck: Differential equations; Theory of analytic functions; Integral equations—By Professor A. Dresden: Calculus of variations.—By Professor C. S. Slichter: Mechanics.—By Professor L. W. Dowling: Projective geometry.

YALE UNIVERSITY.—By Professor J. K. Whittemore: Differential geometry; Special topics in advanced differential geometry.—By Professor W. A. Wilson: Functions of a real variable; Special topics in the theory of aggregates.—By Professor E. J. Miles: Advanced calculus of variations. —By Professor E. W. Brown: Celestial mechanics.—By Professor J. I. Tracey: Higher algebra.—By Professor James Pierpont: Non-euclidean geometry and Einstein's theory.—By Mr. Mikesh: Teachers' course.

Drury College, on the occasion of its fiftieth anniversary in June, 1923, conferred the honorary degree of doctor of laws on Professor B. F. Finkel, of the department of mathematics, founder of the AMERICAN MATHEMATICAL MONTHLY.

Professor G. D. Birkhoff of Harvard University received the honorary degree of doctor of science from Brown University at the June, 1923, commencement.

The following Italian professors have been transferred as indicated: Professor G. Armellini from the chair of higher mechanics at Pisa to that of astronomy at Rome; Professor A. Comessati from the chair of analytic geometry at Cagliari to that of descriptive geometry at Padua; Professor E. Laura from the chair of rational mechanics at Pavia to the same chair at Padua; Professor A. Palatini from the chair of rational mechanics at Messina to that of mathematical physics at Naples.

The following have been appointed to associate professorships: Dr. E. Bompiani, in analytic geometry, at the Milan Technical School; Dr. C. Rosati, in projective and descriptive geometry, at the University of Pisa; Dr. G. Sannia, in analytic geometry, and Dr. G. Vitali, in infinitesimal calculus, at the University of Modena.

Professor G. D. Olds of Amherst has been elected acting President of that college for the academic year 1923–1924, with the understanding that he will be made president at the end of the year. Professor Olds has been professor of mathematics at Amherst since 1891, was dean from 1908 to 1920, and acting president 1920–1921.

Associate Professor R. C. Archibald of Brown University has been promoted to a full professorship of mathematics.

Professor P. J. Daniell, of Rice Institute, has been appointed to the Town Trust chair of mathematics at the University of Sheffield.

Captain D. M. Garrison, of the corps of Professors of Mathematics, U. S. Navy, is retiring from the Navy, and has accepted the professorship of mathematics at St. John's College, Annapolis, Md. Captain Garrison has been a member of the department of mathematics at the U. S. Naval Academy for twenty years, and for the past five years, during the difficult period of readjustment following the World War, has been head of the department. He will be succeeded as head of the department by Commander A. J. Chantry, Jr., of the corps of Naval Constructors.

Mr. P. Y. Yoder, of the University of Kansas, has been appointed professor of mathematics and physics at Blue Ridge College, Maryland.

Professor J. V. DePorte, of the New York State College for Teachers, has been granted leave of absence for the academic year 1923–24, which he will spend at Johns Hopkins University.

Dr. C. H. Yeaton and Dr. F. E. Carr have been appointed assistant professors of mathematics at Oberlin College.

Professor C. N. Mills, of Heidelberg University, Tiffin, Ohio, has accepted a professorship of mathematics at the South Dakota State Normal School.

Mr. Norman Anning, instructor in mathematics at the University of Michigan, has been promoted to an assistant professorship.

Dr. Frederick Wood, instructor in mathematics at the University of Wisconsin, has accepted a professorship of mathematics in the State Normal School, Indiana, Pennsylvania.

At the University of Iowa, Assistant Professor W. H. Wilson is on leave of absence for the academic year 1923–1924. Mr. R. E. Kennon, instructor in mathematics, has resigned to accept a position as examiner of the Iowa State Insurance Department.

At the University of Texas, Associate Professor E. L. Dodd has been promoted to a full professorship and made chairman of the department. Associate Professor R. L. Moore has been promoted to a full professorship, Adjunct Professor H. J. Ettlinger to an associate professorship, and Dr. P. M. Batchelder and Miss Mary Decherd to adjunct professorships.

Assistant Professor E. W. Pehrson, of the University of Utah, has been promoted to an associate professorship of mathematics.

The following appointments to instructorships in mathematics are announced: Mr. H. E. Arnold and Mr. G. W. Bain, at Wesleyan University; Mr. J. P. Ballantine, of the University of Chicago, at Columbia University; Dr. I. A. Barnett, of the University of Saskatchewan, at the University of Cincinnati; Mr. H. W. Chandler, of the University of Minnesota, at the University of Florida; Mr. T. H. Milne, of the University of Alberta, at the University of Buffalo; Dr. Jesse Osborn, of Cornell University, at the University of Iowa. Dr. I. Maizlish, of the University of Minnesota, has been appointed instructor in physics at Lehigh University.

Dr. C. A. Garabedian, of Harvard University, has received an appointment as Parker Traveling Fellow for the academic year 1923–1924.

Professor W. Killing died February 11, 1923, at the age of seventy-two years.

Professor S. D. Killam, of the University of Alberta, was drowned in Lake Annis, Nova Scotia, July 22, 1923.

Dr. T. M. Blakslee, of Des Moines College, died January 30, 1923.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

BERZOLARI (L.). Geometria analitica. II: Curve e superficie del secondo ordine. 2a edizione riveduta ed ampliata. (Manuali Hoepli.) Milano, Hoepli, 1922. 10 + 474 pp.

BIEBERBACH (L.). Theorie der Differentialgleichungen. (Die Grundlehren der mathematischen Wissenschaften, Band 6.) Berlin, Springer, 1923. 8 + 320 pp.

BREWSTER (G. W.). Common sense of the calculus. Oxford, Clarendon Press, 1923. 62 pp.

BROAD (C. D.). Scientific thought. London, Kegan Paul, and New York, Harcourt, Brace and Company, 1923. 555 pp.

BROWNLEE (J.). Log $\Gamma(x)$ from $x = 1$ to 50.9 by intervals of .01. (Tracts for Computers, No. 9.) London, Cambridge University Press, 1923. 23 pp.

DE COMBEROUSSE (C.). Cours de mathématiques. Tome 4: Algèbre supérieure. 2e partie, entièrement refondue par R. de Montessus de Ballore. Paris, Gauthier-Villars, 1923. 16 + 1033 pp.

DARBOUX (G.). Leçons sur la théorie des surfaces. Partie 3: Lignes géodésiques. Paramètres différentiels. Déformation des surfaces. Nouveau tirage. Paris, Gauthier-Villars, 1923.

DICKSON (L. E.). Algebras and their arithmetics. Chicago, University of Chicago Press, 1923. 241 pp. $2.35

DICKSON (L. E.), MITCHELL (H. H.), VANDIVER (H. S.), and WAHLIN (G. E.). Algebraic numbers. (Bulletin of the National Research Council.) Washington, National Academy of Sciences, 1923. 96 pp. $1.50

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band III 3, Heft 6: R. Weitzenböck, Neuere Arbeiten der algebraischen Invariantentheorie. Differentialinvarianten. Leipzig, Teubner, 1922.

EUDOXE. Géométrie pure et géométrie synthétique. Paris, A. Blanchard, 1923. 8vo. 80 pp.

FUETER (R.). Synthetische Zahlentheorie. Neue Ausgabe. Berlin, Vereinigung wissenschaftlicher Verleger, 1921. 8 + 271 pp.

GAUSS (C. F.). Werke. Herausgegeben von der Gesellschaft der Wissenschaften zu Göttingen. Band 10, Abteilung 2, 1te, 4te und 5te Abhandlung. Leipzig, Teubner, 1922–23.

HAAG (J.). Cours complet de mathématiques spéciales. Tome 4: Géométrie descriptive. Trigonométrie. Paris, Gauthier-Villars, 1923. 11 + 152 pp.

HEFFTER (L.). Lehrbuch der analytischen Geometrie. Band 2: Geometrie im Bündel und im Raum. Leipzig, Teubner, 1923. 12 + 423 pp.

KNESER (A.). Die Integralgleichungen und ihre Anwendungen in der mathematischen Physik. 2te, umgearbeitete Auflage. Braunschweig, Vieweg, 1922. 8 + 292 pp.

KRULL (W.). Ueber Begleitmatrizen und Elementarteilertheorie. (Diss.) Freiburg i. Br., 1922.

DE LAUNAY (L.). Les grands hommes de France: Descartes. Paris, Payot, 1923.

LONEY (S. L.). The elements of coordinate geometry. Part 2: Trilinear coordinates. London, Macmillan, 1923. 8 + 228 pp.

MITCHELL (H. H.). See DICKSON (L. E.).

DE MONTESSUS DE BALLORE (R.). See DE COMBEROUSSE (C.).

MUIR (T.). The theory of determinants in the historical order of development. Volume 4: The period 1880 to 1900. London, Macmillan, 1923. 31 + 508 pp.

NAGAOKA (H.) and SAKURAI (S.). Tables of theta functions, elliptic integrals K and E, and associated coefficients in the numerical calculation of elliptic functions. (Scientific Papers of the Institute of physical and chemical Research, Volume II.) Tokyo, 1922.

NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS. The reorganization of mathematics in secondary education, a report by the National Committee on Mathematical Requirements under the auspices of the Mathematical Association of America. Hanover, N. H., Mathematical Association of America, 1923. 11 + 652 pp.

NIELSEN (N.). Traité élémentaire des nombres de Bernoulli. Paris, Gauthier-Villars, 1923. 398 pp.

PASCH (M.). Die Begriffswelt des Mathematikers in der Vorhalle der Geometrie. Leipzig, Meiner, 1922.

RIEMANN (B.). Ueber die Hypothesen, welche der Geometrie zu Grunde liegen. Neu herausgegeben und erläutert von H. Weyl. 3te Auflage. Berlin, Springer, 1923. 6 + 48 pp.

SAKURAI (S.). See NAGAOKA (H.).

STUDY (E.). Einleitung in die Theorie der Invarianten linearer Transformationen auf Grund der Vektorenrechnung. 1ter Teil. Braunschweig, Vieweg, 1923. 2 + 268 pp.

TROPFKE (J.). Geschichte der Mathematik. 2te, verbesserte und sehr vermehrte Auflage. Band 4: Ebene Geometrie. Berlin, Vereinigung wissenschaftlicher Verleger, 1923. 238 pp.

DE LA VALLÉE POUSSIN (C.). Cours d'analyse infinitésimale. 4e édition. Tome 2. Paris, Gauthier-Villars, 1922. 15 + 478 pp.

VANDIVER (H. S.). See DICKSON (L. E.).

DE VRIES (J. F.). Analytische behandeling van de rationale kromme van den vierden graad in een vierdimensionale ruimte. s' Gravenhage, Nijhoff, 1922. 11 + 159 pp.

WAHLIN (G. E.). See DICKSON (L. E.).

WEITZENBÖCK (R.). See ENCYKLOPÄDIE.

WEYL (H.). See RIEMANN (B.).

WITTING (A.). Funktionen, Schaubilder und Funktionstafeln. Eine elementare Einführung in die graphische Darstellung und in die Interpolation. Leipzig, Teubner, 1922.

PART II. APPLIED MATHEMATICS

ANNUAIRE pour l'an 1923 publié par le Bureau des Longitudes. Paris, Gauthier-Villars, 1923. 872 pp.

ARISTOTLE. The works of Aristotle translated into English. De Cælo, by J. L. Stocks. Oxford, University Press, 1922.

BAUER (E.). La théorie de la relativité. Paris, Librairie de l'Enseignement technique, 1922. 12mo. 4 + 128 pp.

BELL (L.). The telescope. London, McGraw-Hill, 1922. 9 + 287 pp.

BIGELOW (F. H.). Atmospheric radiation, electricity and magnetism. Vienna, 1922. 89 pp.

BJORCK (S.). Le tourbillon cartésien. Ses conséquences. Stockholm, 1922. 8vo. 15 pp.

BOUASSE (H.). La question préalable contre la théorie d'Einstein. Paris, Blanchard, 1923. 28 pp.

TYCHONIS BRAHE opera omnia. Tomi quinti fasciculus posterior. Edidit I. L. E. DREYER. Hauniæ, Libraria Gyldendaliana, 1923. 127 pp.

BROSE (H. L.). See SOMMERFELD (A.).

CAMPBELL (N. R.). Modern electrical theory. Supplementary chapters. Chapter 16: Relativity. Cambridge, University Press, 1923. 8 + 116 pp.

CASAZZA (G.). I principî della meccanica alla luce della critica. Roma, Albrighi, Segati e Cia., 1921. 174 pp.

CHRISTESCO (S.). La lumière relative et l'expérience de Michelson. Paris, Alcan, 1923. 8vo. 48 pp.

DUBROCA (M.). L'erreur de M. Einstein. L'inacceptable théorie. L'éther et le principe de la relativité. Paris, Gauthier-Villars, 1922. 8vo. 48 pp.

EDDINGTON (A. S.). The mathematical theory of relativity. Cambridge, University Press, 1923. 10 + 248 pp.

ENNIS (W. D.). Thermodynamics, abridged. 2d edition, corrected. New York, 1922. 244 pp. $4.00

FLEMING (J. A.). Waves and ripples on water, air and æther. 4th issue, revised. London, Sheldon Press, and New York, Macmillan, 1923. 12 + 299 pp.

FOURNIER (G.). La relativité vraie et la gravitation universelle. Paris, Gauthier-Villars, 1923. 8 + 130 pp.

FRITSCHE (J.). Die Berechnung des symmetrischen Stockwerkrahmens mit geneigten und lotrechten Ständern mit Hilfe von Differenzengleichungen. Berlin, Springer, 1923. 6 + 90 pp.

GLAZEBROOK (R.). A dictionary of applied physics. Volume 3: Meteorology, metrology and measuring apparatus. Volume 4: Light, sound, radiology. London, Macmillan, 1923. 7 + 839 + 8 + 914 pp.

GÖTZE (R.). See PASCHEN (F.).

GRÄTZ (L.). Die Atomtheorie in ihrer neuesten Entwickelung. Sechs Vorträge. Stuttgart, Engelhorn, 1921. 8 + 93 pp.

HAMMICK (D. L.). See PERRIN (J.).

Hund (A.). Hochfrequenzmesstechnik. Ihre wissenschaftlichen und praktischen Grundlagen. Berlin, Springer, 1922.

Jäger (G.). Theoretische Physik. II–III: 5te Auflage. Berlin, Vereinigung wissenschaftlicher Verleger, 1921–22.

König (A.). Die Fernrohre und Entfernungsmesser. Berlin, Springer, 1923. 8vo. 7 + 207 pp.

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Meetings of the Society have been fixed at the following times and places:

New York City, October 27, 1923.

Abstracts must be in the hands of the Secretary of the Society not later than October 13.

The Southwestern Section, in Columbia, Missouri, December 1, 1923. See also page 378.

Abstracts must be in the hands of the Secretary of the Section, E. B. Stouffer, not later than November 16.

The Annual Meeting of the Society, in New York City, December 27–28, 1923. See also this Bulletin, vol. 29, No. 4 (April, 1923), p. 188.

Abstracts must be in the hands of the Secretary of the Society not later than November 29.

The Twentieth Western Meeting of the Society (Fifty-second Meeting of the Chicago Section), in Cincinnati, December 28–29, 1923. See also this Bulletin, vol. 29, No. 5 (May, 1923), p. 197.

Abstracts must be in the hands of the Secretary of the Chicago Section, Arnold Dresden, 2114 Vilas St., Madison, Wis., not later than November 29.

R. G. D. Richardson,
Secretary of the Society.

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Any communication intended for this BULLETIN may be addressed to the American Mathematical Society, Prince and Lemon Streets, Lancaster, Pa., or may be sent to separate officials of the Society as follows:

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, 304 Hicks Ave., Columbia, Mo. Reviews should be sent to J. W. YOUNG, Dartmouth College, Hanover, N. H. Notes should be sent to R. W. BURGESS, Brown University, Providence, R. I.

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